

LEAVITT PATH ALGEBRAS OF SEPARATED GRAPHS

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ABSTRACT. The construction of the Leavitt path algebra associated to a directed graph E is extended to incorporate a family C consisting of partitions of the sets of edges emanating from the vertices of E . The new algebras, $L_K(E, C)$, are analyzed in terms of their homology, ideal theory, and K-theory. These algebras are proved to be hereditary, and it is shown that any conical abelian monoid occurs as the monoid $\mathcal{V}(L_K(E, C))$ of isomorphism classes of finitely generated projective modules over one of these algebras. The lattice of trace ideals of $L_K(E, C)$ is determined by graph-theoretic data, namely as a lattice of certain pairs consisting of a subset of E^0 and a subset of C . Necessary conditions for $\mathcal{V}(L_K(E, C))$ to be a refinement monoid are developed, together with a construction that embeds (E, C) in a separated graph (E_+, C^+) such that $\mathcal{V}(L_K(E_+, C^+))$ has refinement.

1. INTRODUCTION

Leavitt introduced in [30] a class of algebras $L_K(m, n)$ (in current notation), for $1 \leq m \leq n$, over an arbitrary field K , which have a universal isomorphism between the free modules of ranks m and n . Some years later and independently, Cuntz constructed and studied in [18], [19] the class of C^* -algebras \mathcal{O}_n nowadays known as *Cuntz algebras*, which are generated by n isometries S_1, \dots, S_n such that $\sum_{i=1}^n S_i S_i^* = 1$. It turns out that the Leavitt algebra $L_{\mathbb{C}}(1, n)$ of type $(1, n)$ is isomorphic to a dense $*$ -subalgebra of the Cuntz algebra \mathcal{O}_n , and both algebras are simple and purely infinite for $n \geq 2$.

Cuntz and Krieger [20] generalized the construction of the Cuntz algebras \mathcal{O}_n by considering a class of C^* -algebras associated to finite square matrices with entries in $\{0, 1\}$, and established an important connection between these *Cuntz-Krieger algebras* and the theory of dynamical systems. Subsequently, it was realized that the Cuntz-Krieger algebras are specific cases of a more general structure, the *graph C^* -algebras* $C^*(E)$ initially studied in depth in [29]. We refer the reader to [36] for further information on this important class of C^* -algebras. The C^* -analogs $U_{(m,n)}^{\text{nc}}$ of the Leavitt algebras $L_K(m, n)$ for $1 < m \leq n$ were studied by Brown

2000 *Mathematics Subject Classification.* Primary 16D70; Secondary 46L35, 06F05, 16E20, 16G20, 16S10, 20M14.

Key words and phrases. Graph algebra, Leavitt path algebra, separative cancellation, refinement monoid, nonstable K-theory, V-monoid, ideal lattice, trace ideal.

The first-named author was partially supported by DGI MICIIN-FEDER MTM2008-06201-C02-01, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. A substantial part of this work was carried out during a research stay of the first-named author at the Department of Mathematics of UCSB in 2009, supported by the grant PR2008 0250 from the Ministerio de Ciencia e Innovación, Spain.

[16] and McClanahan [31] in the case $n = m$ and by McClanahan [32] in general. However, no description in terms of graphs has been developed so far for this class of C^* -algebras.

Leavitt path algebras $L_K(E)$, natural algebraic versions of graph C^* -algebras, were introduced and studied firstly in [1] and [9] for row-finite graphs, and in [2] and [39] for general graphs; see also [4], [5], [10], [11], [26]. These algebras generalize the classical Leavitt algebras of type $(1, n)$ in much the same way as graph C^* -algebras generalize the Cuntz algebras \mathcal{O}_n . The Leavitt path algebra of a directed graph $E = (E^0, E^1, r, s)$ is obtained from E by first adding a copy of E^1 , written $E^{1*} = \{e^* \mid e \in E^1\}$, and extending r and s so that $r(e^*) = s(e)$ and $s(e^*) = r(e)$. The Leavitt path algebra over a field K , $L_K(E)$, is then defined to be the quotient of the ordinary path algebra of $(E^0, E^1 \sqcup E^{1*}, r, s)$ obtained by imposing the Cuntz-Krieger relations. The new algebras we introduce are also quotients of the path algebra of $(E^0, E^1 \sqcup E^{1*}, r, s)$, but with Cuntz-Krieger relations only imposed on selected sets of edges.

The present paper initiates the study of much larger classes of algebras and C^* -algebras based on the concept of a *separated graph* (E, C) , namely a directed graph E together with a family C that gives partitions of the set of edges departing from each vertex of E . The associated Leavitt path algebra $L_K(E, C)$ and graph C^* -algebra $C^*(E, C)$ incorporate the existing versions $L_K(E)$ and $C^*(E)$ for a particular choice of C on the one hand, and many algebras of classical Leavitt and Cuntz type on the other. As an immediate indicator of the broader generality of the new classes of algebras, we mention the following examples. First, any free product of algebras $L_K(1, n)$ (or even $L_K(1, \aleph)$, where \aleph is an arbitrary cardinal) appears as $L_K(E, C)$ for suitable E and C (Proposition 2.10). Second, for any $n \geq m \geq 1$, there is a separated graph (E, C) such that $L_K(E, C) \cong M_{m+1}(L_K(m, n)) \cong M_{n+1}(L_K(m, n))$ and the corner of $L_K(E, C)$ corresponding to one vertex of E is isomorphic to $L_K(m, n)$ itself (Proposition 2.12). Similarly, free products of Cuntz algebras \mathcal{O}_n , and matrix C^* -algebras $M_{m+1}(U_{(m,n)}^{\text{nc}})$, appear as $C^*(E, C)$ for suitable E and C [7].

The construction of the graph C^* -algebra $C^*(E, C)$ of a separated graph (E, C) is given in [7], where we also prove that the Leavitt path algebra $L_{\mathbb{C}}(E, C)$ embeds as a dense $*$ -subalgebra of $C^*(E, C)$. As mentioned above, the C^* -algebras $C^*(E, C)$ enable us to incorporate into the theory the C^* -algebras $U_{(m,n)}^{\text{nc}}$ studied by Brown and McClanahan. Moreover, by using results of Thomsen [38] we will compute the K -theory of these algebras. Also, in a forthcoming paper we will attach a dynamical system to a suitable quotient $\mathcal{O}_{m,n}$ of the C^* -algebra $U_{(m,n)}^{\text{nc}}$, generalizing the classical construction of Cuntz and Krieger for \mathcal{O}_n . Indeed, the C^* -algebra $\mathcal{O}_{m,n}$ is the crossed product corresponding to the universal (m, n) -dynamical system, where an (m, n) -*dynamical system* consists of a compact Hausdorff space Ω , admitting two clopen decompositions $\Omega = \bigsqcup_{i=1}^n U_i = \bigsqcup_{j=1}^m V_j$, together with homeomorphisms

$$\Delta_i: U_i \rightarrow U_1, \quad \Theta_j: V_j \rightarrow V_1, \quad \Phi_{11}: V_1 \rightarrow U_1,$$

for $i = 2, \dots, n$ and $j = 2, \dots, m$.

Our main motivation for the study of the new classes of algebras $L_K(E, C)$ and $C^*(E, C)$ is of a K -theoretical nature. While ordinary Leavitt path algebras constitute quite a large class,

their monoids of finitely generated projective modules satisfy some restrictive conditions, as was proved in [9]. In particular, these monoids are always *separative monoids* and satisfy the *Riesz refinement property* (see below for definitions). For the resolution of some important open problems in both ring theory and C^* -algebra theory, we need a larger class of algebras, whose monoids satisfy the latter property but not the former. The main open problem we wish to address, in the context of C^* -algebras, is the construction of a C^* -algebra of real rank zero containing both finite and properly infinite *full* projections. It is worth to mention that Rørdam has constructed in [37] examples of simple C^* -algebras having both finite and (properly) infinite projections. However, Rørdam's examples do not have real rank zero. The analogous question of whether an *exchange ring* can have both finite and properly infinite full idempotents seems to be wide open. Indeed, such an example would provide a solution to the fundamental *Separativity Problem* for exchange rings of [8]. Note that by [8, Theorem 7.2], the C^* -algebras of real rank zero are exactly the C^* -algebras that happen to be exchange rings, and that for an exchange ring R , the monoid $\mathcal{V}(R)$ is always a refinement monoid [8, Corollary 1.3]. The methods developed in the present paper enable us to construct Leavitt path algebras $L_K(E_+, C^+)$ of separated graphs (E_+, C^+) having finite and properly infinite full idempotents, and such that the monoids $\mathcal{V}(L_K(E_+, C^+))$ satisfy the refinement property. For instance, one can obtain such an algebra $L_K(E_+, C^+)$ by refining the Leavitt path algebra $L_K(E, C)$ attached to the classical Leavitt algebra $L_K(m, n)$, as shown in Example 9.4. Although the algebra $L_K(E_+, C^+)$ is not an exchange ring, we expect that the use of techniques of universal localization, as in [4], will lead to the construction of a class of exchange algebras with the same structure of finitely generated projective modules.

1.1. Contents. We now explain in more detail the contents of this paper. In Section 2, we will define our basic object of study, the Leavitt path algebras $L_K(E, C)$ of separated graphs (E, C) , as well as the corresponding class of *Cohn path algebras* $C_K(E, C)$, generalizing a class of algebras studied by Cohn in [17], and algebras we call *Cohn-Leavitt algebras*. The Cohn path algebras are obtained from the Leavitt path algebras by omitting one of the so-called Cuntz-Krieger relations, and the Cohn-Leavitt algebras interpolate between the two classes by admitting a selection of Cuntz-Krieger relations. (See Definitions 2.2, 2.3, and 2.5 for the precise definitions.) We will denote by $CL_K(E, C, S)$ the Cohn-Leavitt algebra associated to (E, C, S) , where S is a family consisting of some of the finite sets in C . The Cohn-Leavitt algebras not only provide an interesting larger class of algebras that can be analyzed by the same techniques, but they allow us to write all Leavitt path algebras (in fact, all Cohn-Leavitt algebras) as direct limits of algebras based on finite graphs (see Proposition 3.6). The algebras $L_K(E, C)$, $C_K(E, C)$, and $CL_K(E, C, S)$ are homologically well behaved: all are hereditary (in an appropriate nonunital sense when E has infinitely many vertices); moreover, the K -algebra unitizations of these algebras are hereditary in the standard sense (Theorem 3.7 and proof).

Much of our effort is devoted to analyzing the abelian monoids $\mathcal{V}(A)$ associated to the algebras A under consideration. (As recalled below, $\mathcal{V}(A)$ consists of the isomorphism classes of finitely generated projective A -modules, equivalently, the Murray-von Neumann equivalence

classes of idempotent matrices over A .) In the present setting, these monoids are entirely determined by graph-theoretic data, just as in the case of ordinary Leavitt path algebras. We refer to [9] for the latter, where a monoid $M(E)$ was defined for an arbitrary row-finite graph E and shown to be naturally isomorphic to $\mathcal{V}(L_K(E))$. Here, we define analogous monoids $M(E, C, S)$ and construct natural isomorphisms $M(E, C, S) \cong \mathcal{V}(CL_K(E, C, S))$ (Theorem 4.3). (The non-separated case reduces to that of ordinary Leavitt path algebras, and extends the result of [9] to non-row-finite graphs.)

The monoids $M(E, C) \cong \mathcal{V}(L_K(E, C))$ (corresponding to Leavitt path algebras of separated graphs (E, C)) provide a measure of the breadth of this new class of algebras: Every conical abelian monoid is isomorphic to $M(E, C)$ for some (E, C) (Proposition 4.4). Combined with Theorems 3.7 and 4.3, this provides both an extension and a more “visual” version of a result of Bergman and Dicks [15]: Given any conical abelian monoid M , there exists a hereditary K -algebra $A = L_K(E, C)$ such that $\mathcal{V}(A) \cong M$ (Corollary 4.5).

In the case of a non-separated graph E , the monoid $M(E) \cong \mathcal{V}(L_K(E))$ is a refinement monoid, as proved for row-finite graphs in [9] and extended here to arbitrary graphs (Corollary 5.16). We find sufficient conditions for $M(E, C, S)$ to have refinement (Theorem 5.15), and we develop a construction by which a separated graph (E, C) can be embedded in a separated graph (E_+, C^+) such that $M(E_+, C^+)$ has refinement and preserves key properties of $M(E, C)$ (Theorem 8.9). In particular, if $M(E, C)$ does not have separative cancellation ($2x = x + y = 2y \implies x = y$), then $M(E_+, C^+)$ does not have it either. Further, if $M(E, C)$ is simple, we can arrange the construction so that it embeds in the monoid of order-units of $M(E_+, C^+)$ together with zero and so that the latter is a simple, divisible, refinement monoid (Theorem 9.3). Given that $M(E, C)$ can be an arbitrary simple conical abelian monoid, this provides a “visual” version of an embedding theorem of Wehrung [40].

For ordinary Leavitt path algebras $L_K(E)$, the lattice of graded ideals (and even the lattice of all ideals, if condition (K) holds) is isomorphic to a lattice formed from graph-theoretic data [39]. Such a result does not hold for separated graphs, since the algebras $L_K(E, C)$ and $CL_K(E, C, S)$ typically have far more complicated ideal structures than $L_K(E)$. Nonetheless, we can capture the lattice of *trace ideals* of $CL_K(E, C, S)$ (these coincide with the idempotent-generated ideals). This lattice is isomorphic to the lattice of order-ideals of $M(E, C, S)$ (Propositions 6.2 and 10.10) and to a certain lattice of pairs (H, G) where H is a hereditary subset of E^0 and $G \subseteq C$ (Theorem 6.9). Consequently, we derive necessary and sufficient conditions for $M(E, C, S)$ to be simple, equivalently, for $CL_K(E, C, S)$ to be “trace-simple” (Theorems 7.1 and 7.6).

1.2. Recall that for a unital ring R , the monoid $\mathcal{V}(R)$ is usually defined as the set of isomorphism classes $[P]$ of finitely generated projective (left, say) R -modules P , with an addition operation given by $[P] + [Q] = [P \oplus Q]$. For a nonunital version, see Definition 10.8.

For arbitrary rings, $\mathcal{V}(R)$ can also be described in terms of equivalence classes of idempotents from the ring $M_\infty(R)$ of $\omega \times \omega$ matrices over R with finitely many nonzero entries. The equivalence relation is *Murray-von Neumann equivalence*: idempotents $e, f \in M_\infty(R)$ satisfy $e \sim f$ if and only if there exist $x, y \in M_\infty(R)$ such that $xy = e$ and $yx = f$. Write

$[e]$ for the equivalence class of e ; then $\mathcal{V}(R)$ can be identified with the set of these classes. Addition in $\mathcal{V}(R)$ is given by the rule $[e] + [f] = [e \oplus f]$, where $e \oplus f$ denotes the block diagonal matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. With this operation, $\mathcal{V}(R)$ is an abelian monoid, and it is *conical*, meaning that $a + b = 0$ in $\mathcal{V}(R)$ only when $a = b = 0$.

An (abelian) monoid M is said to be a *refinement monoid* if whenever $a + b = c + d$ in M , there exist x, y, z, t in M such that $a = x + y$ and $b = z + t$ while $c = x + z$ and $d = y + t$.

1.3. We will make frequent use of the machinery developed by Bergman in [12] and [13], which enables us to modify algebras in such a way that we have total control on the \mathcal{V} -monoid of the resulting algebra. We repeatedly use one particular construction from [13], so we find it useful to explain here the construction as well as the phrasing of it that we will use later on. Let R be a unital K -algebra and let P and Q be two finitely generated projective left R -modules. Bergman constructs in [13, page 38] a unital K -algebra $S := R\langle i, i^{-1} : \overline{P} \cong \overline{Q} \rangle$, together with an algebra homomorphism $R \rightarrow S$, such that there is a universal isomorphism $i : \overline{P} \rightarrow \overline{Q}$, where $\overline{X} = S \otimes_R X$ for a left R -module X . The universal property of i is expressed as follows. If $R \rightarrow T$ is an algebra homomorphism and $\phi : T \otimes_R P \rightarrow T \otimes_R Q$ is an isomorphism of T -modules, then there is a unique algebra homomorphism $\psi : S \rightarrow T$ such that $\text{id}_T \otimes_S i = \phi$, where T is an S -module via ψ .

We shall refer to the algebra S described above as *the Bergman algebra obtained from R by adjoining a universal isomorphism between P and Q* . By [13, Theorem 5.2], the monoid $\mathcal{V}(S)$ is exactly the quotient monoid of $\mathcal{V}(R)$ modulo the congruence generated by $([P], [Q])$, so that we modify $\mathcal{V}(R)$ by just introducing a single new relation $[P] = [Q]$.

1.4. Throughout the paper, K will denote a field. All graphs in this paper will be directed graphs $E = (E^0, E^1, s, r)$, where E^0 and E^1 are sets, E^0 is nonempty, and s, r denote the source and range maps $E^1 \rightarrow E^0$. We make no finiteness or countability assumptions. Paths of positive length in E are written in the form $\alpha = e_1 e_2 \cdots e_n$ where the $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for $i < n$, while the paths of length zero in E are identified with the vertices in E^0 . The maps s and r are applied to paths in the obvious manner: $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$ for $\alpha = e_1 e_2 \cdots e_n$, while $s(v) = r(v) = v$ for paths v of length zero. We write $\text{Path}(E)$ for the set of all paths in E . This is a partial semigroup, with $\alpha\beta$ defined and equal to the concatenation “ α followed by β ” whenever $r(\alpha) = s(\beta)$. Concatenation with a path of length zero simply absorbs that vertex; e.g., $v\beta = \beta$ for any path β with source vertex v .

The path algebra of E with coefficients in K will be denoted by $P_K(E)$; this is the K -algebra with basis $\text{Path}(E)$ and multiplication induced from the partial multiplication in $\text{Path}(E)$ together with the rule that $\alpha\beta = 0$ for any paths α, β with $r(\alpha) \neq s(\beta)$.

2. THE ALGEBRAS

In this section, we formally introduce separated graphs and the Leavitt, Cohn, and Cohn-Leavitt algebras based on them.

Definition 2.1. A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $s^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v . (In case v is a sink, we take C_v to be the empty family of subsets of $s^{-1}(v)$.)

If all the sets in C are finite, we shall say that (E, C) is a *finitely separated graph*.

The constructions we introduce revert to existing ones in case $C_v = \{s^{-1}(v)\}$ for each non-sink $v \in E^0$. We refer to a *non-separated graph* or a *trivially separated graph* in that situation.

The separating partitions C in the above definition can be viewed in terms of edge-labellings, if desired. On one hand, if $\ell : E^1 \rightarrow A$ is an edge-labelling, the sets $\ell^{-1}(a) \cap s^{-1}(v)$ for $a \in A$ partition $s^{-1}(v)$ for each $v \in E^0$, and the collection of nonempty such sets forms a separating partition C . On the other, given any separating partition C , the map $E^1 \rightarrow C$ sending $e \in E^1$ to the unique set $X \in C$ such that $e \in X$ is an edge-labelling. The notation of partitions is more convenient for the work below than that of edge-labellings; moreover, nothing we do involves any relation between edges departing from different vertices.

We now introduce the first of three algebras based on a separated graph (E, C) . All three are quotients of the path algebra of the *double* of E , that is, the graph \hat{E} obtained from E by adjoining, for each $e \in E^1$, an edge e^* going in the reverse direction of e , that is $s(e^*) = r(e)$ and $r(e^*) = s(e)$. The map $e \mapsto e^*$ from $E^1 \rightarrow \hat{E}^1$ extends, first, to a bijection of $E^0 \sqcup \hat{E}^1$ with itself such that $v^* = v$ for $v \in E^0$ and $(e^*)^* = e$ for $e \in E^1$; second, to an order-reversing bijection of $\text{Path}(\hat{E})$ with itself; and, finally, to a K -algebra involution (i.e., an anti-automorphism of order 2) on $P_K(\hat{E})$. This involution induces involutions on each of the three quotients of $P_K(\hat{E})$ defined below.

Definition 2.2. The *Leavitt path algebra of the separated graph* (E, C) with coefficients in the field K is the K -algebra $L_K(E, C)$ with generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, subject to the following relations:

- (V) $vv' = \delta_{v,v'}v$ for all $v, v' \in E^0$,
- (E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
- (E2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$,
- (SCK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in X$, $X \in C$, and
- (SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \in C_v$, $v \in E^0$.

The path algebra $P_K(\hat{E})$ is the K -algebra with generating set $E^0 \sqcup \hat{E}^1$ and relations (V), (E1), (E2), so $L_K(E, C)$ is the quotient of $P_K(\hat{E})$ obtained by imposing the additional relations (SCK1), (SCK2).

The Leavitt path algebra $L_K(E)$ is just $L_K(E, C)$ where $C_v = \{s^{-1}(v)\}$ if $s^{-1}(v) \neq \emptyset$ and $C_v = \emptyset$ if $s^{-1}(v) = \emptyset$. Despite the great similarity in the definitions, the Leavitt path algebras of separated graphs encompass a much larger class of algebras than Leavitt path algebras of non-separated graphs. For instance, they include free products of Leavitt algebras and algebras closely related to the Leavitt algebras $L_K(n, m)$ (see Propositions 2.10 and 2.12 below).

The Leavitt path algebras $L_K(E, C)$ can be viewed as algebraic analogs of the C^* -algebras of edge-labelled graphs introduced by Duncan in [21].

In [27], Hazrat gives a construction of *weighted Leavitt path algebras* which incorporate all of the Leavitt algebras $L_K(n, m)$. Neither his construction nor ours is a particular case of the other. Hazrat's basic data can be expressed as a separated graph (E, C) such that each of the sets in C is a finite collection of edges with the same source and the same range. However, the Cuntz-Krieger type relations he imposes involve sums of products e^*e' and $e'e^*$ for edges e and e' that may lie in different sets in C . These relations agree with our (SCK1) and (SCK2) only when each vertex of E emits at most one edge.

Before we give some key examples of our construction, we develop a normal form for the elements of $L_K(E, C)$. For this purpose, we will use the Diamond Lemma [14]. We also define and study two other useful constructs: the Cohn path algebra of a separated graph and the Cohn-Leavitt path algebras of separated graphs with distinguished subsets.

Definition 2.3. The *Cohn path algebra of the separated graph* (E, C) with coefficients in the field K is the K -algebra $C_K(E, C)$ with generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, subject only to the relations (V), (E1), (E2) and (SCK1) of Definition 2.2. In other words, $C_K(E, C)$ is the quotient of the path algebra $P_K(\hat{E})$ obtained by imposing only (SCK1).

The C^* -analog of the Cohn path algebra of a non-separated graph is the Toeplitz-Cuntz-Krieger C^* -algebra of the graph, see [23, Theorem 4.1].

We will give K -bases of $C_K(E, C)$ and $L_K(E, C)$ by specifying particular sets of paths in \hat{E} , that is, by finding vector space sections of the canonical surjections $P_K(\hat{E}) \rightarrow C_K(E, C)$ and $P_K(\hat{E}) \rightarrow L_K(E, C)$. The choice is canonical in the case of the Cohn algebra and depends on the choice of an edge in each finite $X \in C$ in the case of the Leavitt algebra.

For two paths $\gamma, \mu \in \text{Path}(E)$ of positive length, with $s(\gamma) = s(\mu) = v \in E^0$, we say that γ and μ are C -separated if the initial edges of γ and μ belong to different sets $X, Y \in C_v$.

Proposition 2.4. Let (E, C) be a separated graph. Then the set \mathcal{B} of those paths of the form

$$(2.1) \quad \lambda_1 \nu_1^* \lambda_2 \nu_2^* \cdots \lambda_r \nu_r^*, \quad \lambda_i, \nu_i \in \text{Path}(E), \quad r \geq 1,$$

such that ν_i and λ_{i+1} are C -separated paths for each $i = 1, \dots, r-1$, is a K -basis of $C_K(E, C)$. In particular, ν_1, \dots, ν_{r-1} and $\lambda_2, \dots, \lambda_r$ must have positive length when $r > 1$. However, λ_1 and ν_r are allowed to have length zero.

Proof. Let A be the vector space with basis \mathcal{B} . We define a binary product on $\mathcal{B} \sqcup \{0\}$ by the following formula: If $b = \lambda_1 \nu_1^* \cdots \lambda_r \nu_r^*$ and $b' = \gamma_1 \mu_1^* \cdots \gamma_s \mu_s^*$, then

$$bb' := \begin{cases} \lambda_1 \nu_1^* \cdots \lambda_r (\nu_r')^* \gamma_1' \mu_1^* \cdots \gamma_s \mu_s^* & \text{if } \nu_r = \tau \nu_r' \text{ and } \gamma_1 = \tau \gamma_1' \text{ with } \nu_r' \text{ and } \gamma_1' \\ & \text{being } C\text{-separated} \\ \lambda_1 \nu_1^* \cdots (\lambda_r \gamma_1') \mu_1^* \cdots \gamma_s \mu_s^* & \text{if } \gamma_1 = \nu_r \gamma_1' \text{ for some } \gamma_1' \in \text{Path}(E) \\ \lambda_1 \nu_1^* \cdots \lambda_r (\mu_1 \nu_r')^* \cdots \gamma_s \mu_s^* & \text{if } \nu_r = \gamma_1 \nu_r' \text{ for some } \nu_r' \in \text{Path}(E) \\ 0 & \text{otherwise.} \end{cases}$$

It is readily seen that with this product, $\mathcal{B} \sqcup \{0\}$ becomes a semigroup. Extending this product linearly gives a structure of associative algebra to A . It is clear that, with this structure, A is isomorphic to $C_K(E, C)$. This shows the result. \square

We next introduce the “mixed case” of Cohn-Leavitt algebras.

Definition 2.5. Let (E, C) be a separated graph. Let us denote by C_{fin} the subset of C consisting of those X such that $|X| < \infty$, and let S be any subset of C_{fin} . Then let $CL_K(E, C, S)$ be the K -algebra with generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, subject to the relations (V), (E1), (E2) and (SCK1) of Definition 2.2 together with the relations (SCK2) for the sets $X \in S$. Observe that $CL_K(E, C, \emptyset) = C_K(E, C)$ and $CL_K(E, C, C_{\text{fin}}) = L_K(E, C)$. We call $CL_K(E, C, S)$ the *Cohn-Leavitt algebra of the triple (E, C, S)* .

The C^* -analog of Cohn-Leavitt path algebras, for a non-separated graph E and a subset V of regular vertices of E , is introduced in [33, Definition 3.5] under the name of *relative graph algebra*. It is shown in [33, Theorem 3.7] that the relative graph C^* -algebra $C^*(E, V)$ is canonically isomorphic to the graph C^* -algebra $C^*(E_V)$ of a suitable graph E_V .

We are now in a position to describe a basis of $CL_K(E, C, S)$, and so, in particular, of $L_K(E, C)$.

Definition 2.6. Let (E, C) be a separated graph and $S \subseteq C_{\text{fin}}$. For each $X \in S$ we select an edge $e_X \in X$. Let γ and ν be paths in E such that $r(\gamma) = r(\nu)$ and $|\gamma| > 0$, $|\nu| > 0$. Let α and β be the terminal edges of γ and ν respectively. Then the path $\gamma\nu^*$ in \hat{E} is said to be *reduced with respect to S* in case for every $X \in S$ we have $(\alpha, \beta) \neq (e_X, e_X)$. Moreover, $\gamma\nu^*$ is called *reduced with respect to S* in case either γ or ν has length zero, i.e., all real and ghost paths (including the trivial ones) are automatically reduced with respect to any subset S of C_{fin} .

A path p as in (2.1), such that ν_i and λ_{i+1} are C -separated paths for each $i = 1, \dots, r-1$, is said to be *reduced with respect to S* in case $\lambda_i\nu_i^*$ is reduced with respect to S for each $i = 1, \dots, r$.

Theorem 2.7. *Let (E, C) be a separated graph and let S be a subset of C_{fin} . Then a K -basis of $CL_K(E, C, S)$ is given by the family \mathcal{B}' consisting of all the paths in the set \mathcal{B} described in Proposition 2.4 which are reduced with respect to S .*

Proof. Let $CL_K(E, C, S) \oplus K \cdot \tilde{1}$ be the formal unitization of $CL_K(E, C, S)$, where $\tilde{1}$ is a new identity element. We use Bergman’s version of the Diamond Lemma [14, Lemma 1.1, Theorem 1.2] to show that $\mathcal{B}' \sqcup \{\tilde{1}\}$ is a basis for $CL_K(E, C, S) \oplus K \cdot \tilde{1}$, from which the theorem follows.

Let W be the free abelian monoid on the set $E^0 \sqcup E^1 \sqcup (E^1)^* = E^0 \sqcup \hat{E}^1$. For $X \in S$, choose $e_X \in X$ as in Definition 2.6, and set $X' := X \setminus \{e_X\}$. Define a weight function $\text{wt} : W \rightarrow \mathbb{Z}^+$ so that

$$\begin{aligned} \text{wt}(v) &= 1 \text{ for all } v \in E^0, \\ \text{wt}(e_X) &= 2 \text{ for all } X \in S, \end{aligned}$$

$\text{wt}(f) = 1$ for all other $f \in \hat{E}^1$,

and so that the weight of any word is the sum of the weights of its letters. (In particular, $\text{wt}(1_W) = 0$.) Then define a partial order \leq on W by the following rule:

$$a \leq b \iff a = b \text{ or } \text{wt}(a) < \text{wt}(b).$$

Observe that \leq is a semigroup ordering on W , and that it satisfies the descending chain condition.

Next, let F denote the monoid algebra $K[W]$, that is, the free unital K -algebra on $E^0 \sqcup \hat{E}^1$, and let \mathcal{S} be the reduction system in F consisting of the following pairs:

- (1) $(vw, \delta_{v,w}v)$ for $v, w \in E^0$,
- (2) $(ve, \delta_{v,s(e)}e)$ for $v \in E^0$ and $e \in \hat{E}^1$,
- (3) $(ew, \delta_{w,r(e)}e)$ for $w \in E^0$ and $e \in \hat{E}^1$,
- (4) $(ef, 0)$ for $e, f \in \hat{E}^1$ with $r(e) \neq s(f)$,
- (5) $(e^*f, \delta_{e,f}r(e))$ for $e, f \in X \in C$,
- (6) $(e_X e_X^*, s(e_X) - \sum_{e \in X'} ee^*)$ for $X \in S$.

Then $CL_K(E, C, S)$ may be presented as F/I where I is the ideal $\langle W_\sigma - f_\sigma \mid (W_\sigma, f_\sigma) \in \mathcal{S} \rangle$ of F . Some of these reductions are redundant as far as generating I is concerned, namely the cases $(ab, 0)$ of (2), (3), (4). However, these reductions are needed in order to resolve certain ambiguities. Observe that the partial order \leq on W is compatible with \mathcal{S} . (The assignment $\text{wt}(e_X) = 2$ ensures compatibility with the reductions (6).)

In order to apply the Diamond Lemma, we must verify that all ambiguities of \mathcal{S} are resolvable. Since the first terms W_σ of the pairs $(W_\sigma, f_\sigma) \in \mathcal{S}$ all have length 2 (as words in W), there are no inclusion ambiguities. There are four families of overlap ambiguities, corresponding to certain products of the following types:

- (a) $vw x$, for $v, w, x \in E^0$,
- (b) vwe, vew, evw , for $v, w \in E^0$ and $e \in \hat{E}^1$,
- (c) vef, evf, efv , for $v \in E^0$ and $e, f \in \hat{E}^1$,
- (d) efg , for $e, f, g \in \hat{E}^1$.

For instance, a product vef as in (c) is an ambiguity only if $r(e) \neq s(f)$, or $e^*, f \in X$ for some $X \in C$, or $(e, f) = (e_X, e_X^*)$ for some $X \in C$. We indicate resolutions for various cases of these ambiguities, leaving the others to the reader. Reductions will be denoted by \mapsto .

The resolution of an overlap ambiguity of type (a) can be given as follows:

$$\begin{aligned} (vw)x &\mapsto (\delta_{v,w}w)x \mapsto \delta_{v,w}\delta_{w,x}w \\ v(wx) &\mapsto v(\delta_{w,x}w) \mapsto \delta_{w,x}\delta_{v,w}w. \end{aligned}$$

Those of type (b) are resolved in the same manner.

For overlaps of type (c), assume first that $r(e) \neq s(f)$. We then have

$$\begin{aligned} (ve)f &\mapsto (\delta_{v,s(e)}e)f \mapsto 0 \\ v(ef) &\mapsto v(0) = 0, \end{aligned}$$

and similarly for the cases evf and efv . The case evf with $r(e) = s(f)$ resolves trivially. Otherwise, we only have ambiguities to resolve for ve^*f and e^*fv with $e, f \in X \in C$, and for $ve_X e_X^*$ and $e_X e_X^* v$ with $X \in S$. The case ve^*f resolves as

$$\begin{aligned} (ve^*)f &\mapsto (\delta_{v,r(e)}e^*)f \mapsto \delta_{v,r(e)}\delta_{e,f}r(e) \\ v(e^*f) &\mapsto v(\delta_{e,f}r(e)) \mapsto \delta_{e,f}\delta_{v,r(e)}r(e), \end{aligned}$$

and the case e^*fv is similar. Now consider $ve_X e_X^*$ for $v \in E^0$ and $X \in S$, say $X \in C_w$. On one hand,

$$(ve_X)e_X^* \mapsto (\delta_{v,w}e_X)e_X^* \mapsto \delta_{v,w}(w - \sum_{e \in X'} ee^*).$$

On the other hand,

$$v(e_X e_X^*) \mapsto v(w - \sum_{e \in X'} ee^*) \mapsto (|X| \text{ reductions}) \mapsto \delta_{v,w}(w - \sum_{e \in X'} ee^*),$$

since $vw \mapsto \delta_{v,w}w$ and $vee^* \mapsto \delta_{v,w}ee^*$ for $e \in X'$. The resolution of $e_X e_X^* v$ is similar.

The overlaps of type (d) must be separated according to whether or not $r(e) = s(f)$ and whether or not $r(f) = s(g)$. The cases in which $r(e) \neq s(f)$ and/or $r(f) \neq s(g)$ are resolved in the same manner as those above. For the remaining situation, note that there are no overlap ambiguities abc in which ab and bc are both of type e^*f or both of type $e_X e_X^*$. This leaves only the cases $e_X e_X^* f$ and $f^* e_X e_X^*$ with $f \in X \in S$, say $X \in C_v$. We give the resolution of $e_X e_X^* f$; that for $f^* e_X e_X^*$ is analogous. On one hand,

$$(e_X e_X^*)f \mapsto (v - \sum_{e \in X'} ee^*)f \mapsto f - \sum_{e \in X'} ee^* f \mapsto \dots \mapsto \begin{cases} f - 0 = e_X & (\text{if } f = e_X) \\ f - ff^*f \mapsto 0 & (\text{if } f \neq e_X). \end{cases}$$

Thus, $(e_X e_X^*)f$ reduces to $\delta_{f,e_X}e_X$. Since also

$$e_X(e_X^* f) \mapsto e_X(\delta_{e_X,f}r(e_X)) \mapsto \delta_{f,e_X}e_X,$$

this ambiguity is resolved.

Thus, all ambiguities of \mathcal{S} are resolvable. Hence, the cosets of the irreducible words in W form a basis for $CL_K(E, C, S) \oplus K \cdot \tilde{1}$. It only remains to show that the irreducible words are precisely the elements of $\mathcal{B}' \sqcup \{\tilde{1}\}$. That the elements of the latter set are irreducible is clear. Conversely, let $w = a_1 a_2 \dots a_n$ be an irreducible word in W , where the $a_i \in E^0 \sqcup \hat{E}^1$. We cannot have any $a_i \in E^0$ unless $n = 0, 1$, in which case either $w = \tilde{1}$ or $w = a_1 \in E^0$. Now assume that $n \geq 2$ and that all $a_i \in \hat{E}^1$. For $i = 1, \dots, n-1$, we must have $r(a_i) = s(a_{i+1})$, and we cannot have either $(a_i, a_{i+1}) = (e^*, f)$ with $e, f \in X \in C$ or $(a_i, a_{i+1}) = (e_X, e_X^*)$ with $X \in S$. From this, we conclude that $w \in \mathcal{B}'$, as desired, and the proof is complete. \square

When $S = C_{\text{fin}}$ we simply refer to the elements of \mathcal{B}' as the *reduced elements* of \mathcal{B} . As $L_K(E, C) = CL_K(E, C, C_{\text{fin}})$, we immediately get:

Corollary 2.8. *Let \mathcal{B} be the canonical basis of $C_K(E, C)$ given in Proposition 2.4. Then the set \mathcal{B}' of all the reduced elements of \mathcal{B} is a K -basis for $L_K(E, C)$.*

Remark 2.9. Note that the fact that K is a field does not play any role in the determination of the bases \mathcal{B} and \mathcal{B}' of the algebras $C_K(E, C)$ and $L_K(E, C)$ respectively, so that the same arguments would show that the corresponding R -algebras $C_R(E, C)$ and $L_R(E, C)$ are free as R -modules, for any nonzero unital commutative ring R .

We are ready now to give some concrete examples. In the first place, we analyse the algebras corresponding to separated graphs with only one vertex. For a cardinal number \aleph , denote by $L_K(\aleph)$ the Leavitt algebra of type $(1, \aleph)$, namely, the Leavitt path algebra of the (non-separated) graph with one vertex and \aleph edges. (For $\aleph = 1$, $L_K(1) = K[t, t^{-1}]$.)

Proposition 2.10. *Assume that (E, C) is a separated graph and that $|E^0| = 1$. Then we have*

$$L_K(E, C) \cong \ast_{X \in C} L_K(|X|),$$

that is, $L_K(E, C)$ is a free product over K of Leavitt path algebras of type $(1, |X|)$, for $X \in C$.

Proof. Denote by L the free product $\ast_{X \in C} L_K(|X|)$. To keep the different algebras $L_K(|X|)$ apart (since many sets in C may have the same cardinality), identify the copy of $L_K(|X|)$ corresponding to a set $X \in C$ with the Leavitt path algebra $L_K(E_X)$ where E_X is the subgraph of E with $E_X^0 = E^0$ and $E_X^1 = X$. Each inclusion map $E_X \rightarrow E$ induces a K -algebra homomorphism $L_K(|X|) \rightarrow L_K(E, C)$, and the family of these homomorphisms extends uniquely to a unital K -algebra homomorphism $\varphi : L \rightarrow L_K(E, C)$.

Each $e \in E^1$ belongs to a unique $X \in C$, and the symbols e and e^* represent elements of both $L_K(E, C)$ and $L_K(|X|)$. Let \bar{e} and \bar{e}^* denote the elements corresponding to e and e^* in the canonical copy of $L_K(|X|)$ inside L . The elements $1 \in L$ and \bar{e}, \bar{e}^* for $e \in E^1$ satisfy the defining relations of $L_K(E, C)$, so there is a unique unital K -algebra homomorphism $\psi : L_K(E, C) \rightarrow L$ sending $e \mapsto \bar{e}$ and $e^* \mapsto \bar{e}^*$ for all $e \in E^1$. It is clear that $\varphi\psi$ and $\psi\varphi$ are identity maps. This shows the result. \square

Example 2.11. We consider now a key class of examples, the separated graphs which correspond to the Leavitt algebras $L_K(m, n)$ for $1 \leq m \leq n$. Indeed we can think of these Leavitt path algebras as versions of $L_K(m, n)$ which are generated by “partial isometries”. Let us consider the separated graph $(E(m, n), C(m, n))$, where

- (1) $E(m, n)^0 := \{v, w\}$ (with $v \neq w$).
- (2) $E(m, n)^1 := \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ (with $n + m$ distinct edges).
- (3) $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j .
- (4) $C(m, n) = C(m, n)_v := \{\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_m\}\}$.

We will show that the structure of $A_{m,n} := L_K(E(m, n), C(m, n))$ is closely related to the structure of the classical Leavitt algebra $L_K(m, n)$.

Observe that the corner algebras $vA_{m,n}v$ and $wA_{m,n}w$ are full corners of $A_{m,n}$, that is, $A_{m,n}vA_{m,n} = A_{m,n}wA_{m,n} = A_{m,n}$. In particular, $vA_{m,n}v$, $wA_{m,n}w$, and $A_{m,n}$ are Morita equivalent to each other. We will describe their structure below.

Recall that $L_K(m, n)$ is generated by elements X_{ij} and X_{ij}^* , for $i = 1, \dots, m$ and $j = 1, \dots, n$, such that $XX^* = I_m$ and $X^*X = I_n$, where X denotes the $m \times n$ matrix (X_{ij}) and X^* denotes the $*$ -transpose of X , i.e., the $n \times m$ matrix with entries $(X^*)_{ji} = X_{ij}^*$.

The isomorphism $wA_{m,n}w \cong L_K(m, n)$ below was discovered by Pardo [35]. We thank him for permission to give it here.

Proposition 2.12. *Let $m \leq n$ be positive integers, and define $E(m, n)$, $C(m, n)$, $A_{m,n}$ as above.*

- (1) *There are K -algebra isomorphisms*

$$\begin{aligned} A_{m,n} &\cong M_{m+1}(L_K(m, n)) \cong M_{n+1}(L_K(m, n)) \\ vA_{m,n}v &\cong M_m(L_K(m, n)) \cong M_n(L_K(m, n)) & wA_{m,n}w &\cong L_K(m, n). \end{aligned}$$

- (2) *The monoids $\mathcal{V}(L_K(m, n))$, $\mathcal{V}(A_{m,n})$, $\mathcal{V}(vA_{m,n}v)$, and $\mathcal{V}(wA_{m,n}w)$ are all of the form $\langle x \mid mx = nx \rangle$, where the generator x corresponds to the classes $[1]$, $[w]$, $[\alpha_1\alpha_1^*]$, $[w]$ in the four respective cases.*
- (3) *Moreover, $vA_{m,n}v$ is the Bergman algebra obtained from $R := M_n(K) * M_m(K)$ by adjoining a universal isomorphism between the left R -modules Rf_{11} and Rg_{11} , where $(f_{ij})_{i,j=1}^n$ and $(g_{ij})_{i,j=1}^m$ are sets of matrix units corresponding to the factors $M_n(K)$ and $M_m(K)$ respectively.*
- (4) *There is a surjective unital K -algebra homomorphism $\rho : L_K(m, n) \rightarrow vA_{m,n}v$. If $\mathcal{V}(L_K(m, n))$ and $\mathcal{V}(vA_{m,n}v)$ are identified with $\langle x \mid mx = nx \rangle$ as in (2), then $\mathcal{V}(\rho)$ is given by multiplication by m (equivalently, multiplication by n).*

Proof. Set $A := A_{m,n}$ and $L := L_K(m, n)$. We construct various K -algebra homomorphisms between algebras presented by generators and relations. In all cases, it is routine to check that the appropriate relations are satisfied by the proposed images for the generators, and we omit these details.

(1) We identify L with the diagonal copies of itself in the various matrix algebras $M_d(L)$. Let $(e_{ij})_{i,j=1}^{m+1}$ be the standard family of matrix units in $M_{m+1}(L)$, and observe that $M_{m+1}(L)$ is presented by the generators e_{ij} , X_{ij} , X_{ij}^* together with the following three types of relations:

- (a) The defining relations for the X_{ij} and X_{ij}^* in L .
- (b) The matrix unit relations for the e_{ij} .
- (c) The commutation relations $e_{kl}X_{ij} = X_{ij}e_{kl}$ and $e_{kl}X_{ij}^* = X_{ij}^*e_{kl}$ for all i, j, k, l .

Moreover, $M_{m+1}(L)$ is a free left (or right) L -module with basis $\{e_{ij} \mid 1 \leq i, j \leq m+1\}$. Analogous statements hold for $M_m(L)$, and we identify $M_m(L)$ with the corner $eM_{m+1}(L)e$ where $e := e_{11} + \dots + e_{mm}$.

There exist a K -algebra homomorphism $\psi : A \rightarrow M_{m+1}(L)$ such that

$$\begin{aligned} \psi(v) &= e & \psi(w) &= e_{m+1,m+1} \\ \psi(\alpha_i) &= \sum_{l=1}^m X_{li} e_{l,m+1} & \psi(\alpha_i^*) &= \sum_{l=1}^m X_{li}^* e_{m+1,l} & (i = 1, \dots, n) \\ \psi(\beta_j) &= e_{j,m+1} & \psi(\beta_j^*) &= e_{m+1,j} & (j = 1, \dots, m), \end{aligned}$$

and a K -algebra homomorphism $\phi : M_{m+1}(L) \rightarrow A$ such that

$$\begin{aligned} \phi(e_{ij}) &= \beta_i \beta_j^* & (i, j = 1, \dots, m) \\ \phi(e_{i,m+1}) &= \beta_i & (i = 1, \dots, m) \\ \phi(e_{m+1,j}) &= \beta_j^* & (j = 1, \dots, n) \\ \phi(e_{m+1,m+1}) &= w \\ \phi(X_{ij}) &= \beta_i^* \alpha_j + \sum_{l=1}^m \beta_l \beta_i^* \alpha_j \beta_l^* & (i = 1, \dots, m; j = 1, \dots, n) \\ \phi(X_{ij}^*) &= \alpha_j^* \beta_i + \sum_{l=1}^m \beta_l \alpha_j^* \beta_i \beta_l^* & (i = 1, \dots, m; j = 1, \dots, n). \end{aligned}$$

Moreover, ϕ and ψ are mutual inverses. Thus, $A \cong M_{m+1}(L)$.

Isomorphisms between A and $M_{n+1}(L)$ are obtained in a similar fashion, by interchanging the roles of the α_i and β_j in the constructions of ψ and ϕ above.

Since ψ maps v to e , it restricts to an isomorphism of vAv onto $eM_{m+1}(L)e \equiv M_m(L)$. Similarly, $vAv \cong M_n(L)$. Since ψ maps w to $e_{m+1,m+1}$, it restricts to an isomorphism of wAw onto $e_{m+1,m+1}M_{m+1}(L)e_{m+1,m+1}$, and the latter algebra is isomorphic to L .

(2) By [13, Theorem 6.1], $\mathcal{V}(L) = \langle x \mid mx = nx \rangle$ with x corresponding to the class of the free module ${}_L L$, that is, to the class $[1_L]$. Applying the last isomorphism of part (1) immediately yields $\mathcal{V}(wAw) = \langle x \mid mx = nx \rangle$ with x corresponding to $[w]$. In view of the equivalence

$$Aw \otimes_{wAw} (-) : wAw\text{-Mod} \longrightarrow A\text{-Mod},$$

it follows that $\mathcal{V}(A) = \langle x \mid mx = nx \rangle$ with x corresponding to $[w]$. Note that $[w] = [\alpha_1 \alpha_1^*]$ in $\mathcal{V}(A)$ and that $\alpha_1 \alpha_1^* \in vAv$. In view of the equivalence

$$vA \otimes_A (-) : A\text{-Mod} \longrightarrow vAv\text{-Mod},$$

it thus follows that $\mathcal{V}(vAv) = \langle x \mid mx = nx \rangle$ with x corresponding to $[\alpha_1 \alpha_1^*]$.

(3) Let \hat{R} be the Bergman algebra obtained from R by adjoining a universal isomorphism between the modules Rf_{11} and Rg_{11} . Thus, \hat{R} is presented by generators f_{ij} , g_{ij} , u , u^* where

- (a) The f_{ij} satisfy the relations for a complete set of $n \times n$ matrix units.
- (b) The g_{ij} satisfy the relations for a complete set of $m \times m$ matrix units.
- (c) $u = f_{11} u g_{11}$, $u^* = g_{11} u^* f_{11}$, $uu^* = f_{11}$, and $u^*u = g_{11}$.

There is a K -algebra homomorphism $\theta : R \rightarrow vAv$ such that

$$\theta(f_{ij}) = \alpha_i \alpha_j^* \quad \theta(g_{ij}) = \beta_i \beta_j^*$$

for all i, j . The universal property of the Bergman construction implies that θ extends uniquely to a K -algebra homomorphism $\hat{\theta} : \hat{R} \rightarrow vAv$ such that

$$\theta(u) = \alpha_1 \beta_1^* \quad \theta(u^*) = \beta_1 \alpha_1^*.$$

There is a K -algebra homomorphism $\xi : M_m(L) \rightarrow \hat{R}$ such that

$$\begin{aligned} \xi(e_{ij}) &= g_{ij} & (i, j = 1, \dots, m) \\ \xi(X_{ij}) &= \sum_{l=1}^m g_{li} f_{j1} u g_{1l} & (i = 1, \dots, n; j = 1, \dots, m) \\ \xi(X_{ij}^*) &= \sum_{l=1}^m g_{1l} u^* f_{1j} g_{il} & (i = 1, \dots, n; j = 1, \dots, m). \end{aligned}$$

Let $\psi' : vAv \rightarrow M_m(L)$ be the isomorphism obtained by restricting ψ to vAv . Then $\hat{\theta}$ and $\xi\psi'$ are mutual inverses, proving that $\hat{R} \cong vAv$.

(4) There is a K -algebra homomorphism $\rho : L \rightarrow vAv$ such that

$$\rho(X_{ij}) = \alpha_j \beta_i^* \quad \rho(X_{ij}^*) = \beta_i \alpha_j^*$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. Since vAv is generated by the elements $\alpha_j \alpha_i^*, \alpha_j \beta_i^*, \beta_i \alpha_j^*, \beta_i \beta_j^*$, we see that ρ is surjective. As in (2), $\mathcal{V}(L) = \langle x \mid mx = nx \rangle$ with x corresponding to the class $[1]$, and $\mathcal{V}(vAv) = \langle x \mid mx = nx \rangle$ with x corresponding to $[\alpha_1 \alpha_1^*]$. Since $\mathcal{V}(\rho)$ maps $[1]$ to $[v] = \sum_{l=1}^m [\alpha_l \alpha_l^*] = m[\alpha_1 \alpha_1^*]$, we conclude that $\mathcal{V}(\rho)$ is given by multiplication by m . In the given monoid, this is the same as multiplication by n . \square

Remark 2.13. There is a canonical way to define a \mathbb{Z} -graded structure in any $L_K(E, C)$, namely by setting $\deg(e) = 1$ and $\deg(e^*) = -1$ for every $e \in E^1$, and $\deg(v) = 0$ for every $v \in E^0$. However in the particular case of the algebra $A_{m,n} = L_K(E(m, n), C(m, n))$, this \mathbb{Z} -grading does not seem too interesting, since for instance all elements of $vA_{m,n}v$ are homogeneous of degree 0, and thus this grading is not compatible with the natural grading induced by the surjective homomorphism $\rho : L_K(m, n) \rightarrow vA_{m,n}v$ of Proposition 2.12(4), in which the elements $\alpha_j \beta_i^*$ have degree 1 and the elements $\beta_i \alpha_j^*$ have degree -1 . Observe that this \mathbb{Z} -grading coincides with the grading on $vA_{m,n}v$ induced by the \mathbb{Z} -grading on $A_{m,n}$ obtained by setting $\deg(\alpha_i) = 1$, $\deg(\alpha_i^*) = -1$, and $\deg(\beta_j) = \deg(\beta_j^*) = 0$ for all i, j . In terms of the isomorphic algebra \hat{R} , this \mathbb{Z} -grading agrees with the one obtained by giving degree 0 to all elements in $M_n(K) * M_m(K)$ and giving degree 1 to u and degree -1 to u^* .

3. DIRECT LIMITS

In this section we introduce two categories **SSGr** and **SGr** of separated graphs and we study the functoriality and continuity of the constructions in Section 2. In particular, each object of **SSGr** is a direct limit of subobjects based on finite graphs, from which we obtain that

every Cohn-Leavitt algebra is a direct limit of Cohn-Leavitt algebras based on finite graphs. This will be used in the following section in determining the structure of the \mathcal{V} -monoids of Cohn-Leavitt algebras. In the present section, we also prove that every Cohn-Leavitt algebra is hereditary.

Definition 3.1. Define a category **SSGr** of separated graphs with distinguished subsets as follows. The objects of **SSGr** are triples (E, C, S) , where (E, C) is a separated graph and S is a subset of C_{fin} . A morphism $\phi : (E, C, S) \rightarrow (F, D, T)$ in **SSGr** is any graph morphism $\phi : E \rightarrow F$ such that

- (1) ϕ^0 is injective.
- (2) For each $X \in C$ there is a (unique) $Y \in D$ such that ϕ^1 restricts to an injective map $X \rightarrow Y$. Note that the assignment $X \mapsto Y$ defines a set map $\tilde{\phi} : C \rightarrow D$. Moreover, for all $v \in E^0$ and $X \in C_v$, we have $\tilde{\phi}(X) \in D_{\phi^0(v)}$. Hence, $\tilde{\phi}(C_v) \subseteq D_{\phi^0(v)}$.
- (3) The map $\tilde{\phi} : C \rightarrow D$ restricts to a map $S \rightarrow T$ and, for each $X \in S$, ϕ^1 restricts to a bijection $X \rightarrow \tilde{\phi}(X)$.

Definition 3.2. Define a category **SGr** of separated graphs as the full subcategory of **SSGr** whose objects are all triples (E, C, C_{fin}) . When working with the category **SGr** we will use the simplified notation (E, C) instead of (E, C, C_{fin}) , since the third component is determined by the second.

An *SG-subgraph* of a separated graph (F, D) is any separated graph (E, C) such that E is a subgraph of F and the inclusion map $E \rightarrow F$ is a morphism in **SGr**.

Proposition 3.3. *The categories **SSGr** and **SGr** admit arbitrary direct limits. Indeed, given a directed system $\mathcal{D} := \{(E_i, C_i, S_i), f_{ji} \mid i, j \in I, j \geq i\}$ in either category, the underlying sets E^0, E^1, C, S of a direct limit (E, C, S) of this system are just direct limits of the respective systems of sets E_i^0, E_i^1, C_i, S_i in the category of sets. Moreover, E is a direct limit of the system $\{E_i, f_{ji}\}$ in the category of directed graphs.*

Proof. For $l = 0, 1$, let E^l be a direct limit of the directed system $\{E_i^l, f_{ji}^l\}$ in the category of sets, with limit maps $\eta_i^l : E_i^l \rightarrow E^l$. Observe that since all the maps f_{ji}^l are injective, all the maps η_i^0 are injective. For all $j \geq i$ in I , we have $s_{E_j} f_{ji}^1 = f_{ji}^0 s_{E_i}$, whence $\eta_j^0 s_{E_j} f_{ji}^1 = \eta_j^0 f_{ji}^0 s_{E_i} = \eta_i^0 s_{E_i}$. Consequently, there is a unique map $s_E : E^1 \rightarrow E^0$ such that $s_E \eta_i^1 = \eta_i^0 s_{E_i}$ for all i . Likewise, there is a unique map $r_E : E^1 \rightarrow E^0$ such that $r_E \eta_i^1 = \eta_i^0 r_{E_i}$ for all i . Therefore, $E := (E^0, E^1, s_E, r_E)$ is a graph. The pairs $\eta_i := (\eta_i^0, \eta_i^1)$ are graph morphisms, and E together with the η_i is a direct limit of the system $\{E_i, f_{ji}\}$ in the category of directed graphs.

Consider a vertex $w \in E^0$ which is not a sink. Define a relation \sim_w on $s_E^{-1}(w)$ as follows:

$e \sim_w f$ if and only if there exist $i \in I$, $X \in C_i$, and $e', f' \in X$ such that $\eta_i^1(e') = e$ and $\eta_i^1(f') = f$. (Necessarily, $X \in (C_i)_v$ for some $v \in E_i^0$ such that $\eta_i^0(v) = w$.)

Observe that \sim_w is an equivalence relation, and let C_w be the set of all \sim_w -equivalence classes in $s_E^{-1}(w)$.

For any sink $w \in E^0$, set $C_w := \emptyset$. Now set $C := \bigsqcup_{w \in E^0} C_w$; then (E, C) is a separated graph.

Consider $i \in I$ and $X \in C_i$. We claim that there is a unique set $Y \in C$ such that η_i^1 restricts to an injection $X \rightarrow Y$, and that if $X \in S_i$, then $\eta_i^1(X) = Y$.

For $j \geq i$, since f_{ji} is a morphism in **SSGr**, the map f_{ji}^1 restricts to an injection $X \rightarrow \tilde{f}_{ji}(X) \in C_j$. In particular, it follows that η_i^1 restricts to an injection $X \rightarrow E^1$. Now $X \in (C_i)_v$ for some $v \in E_i^0$, and if $w := \eta_i^0(v)$, then $\eta_i^1(e') \sim_w \eta_i^1(f')$ for all $e', f' \in X$. Thus, by definition of C_w , there is a unique $Y \in C_w$ such that $\eta_i^1(X) \subseteq Y$; in fact, $Y = \bigcup_{j \geq i} \eta_j^1(\tilde{f}_{ji}(X))$. In case $X \in S_i$, we have that f_{ji}^1 restricts to a bijection $X \rightarrow \tilde{f}_{ji}(X)$ for all $j \geq i$, from which it follows that $Y = \eta_i^1(X)$. This establishes the claim.

Now set $S := \{\eta_i^1(X) \mid i \in I, X \in S_i\}$. We thus have an object $(E, C, S) \in \mathbf{SSGr}$, and each η_i is a morphism $(E_i, C_i, S_i) \rightarrow (E, C, S)$ in **SSGr**. It is routine to check that (E, C, S) together with the morphisms η_i is a direct limit in **SSGr** for \mathcal{D} . Further, the sets C and S , together with the set maps $\tilde{\eta}_i$, are direct limits for the respective systems $\{C_i, \tilde{f}_{ji}\}$ and $\{S_i, \tilde{f}_{ji}\}$.

Finally, one checks that if all the (E_i, C_i, S_i) are objects in **SGr**, then so is (E, C, S) , and this object together with the morphisms η_i forms a direct limit for \mathcal{D} in **SGr**. \square

Definition 3.4. Let (E, C, S) be an object in **SSGr**. A *complete subobject* of (E, C, S) is an object (F, D, T) of **SSGr** such that F is a subgraph of E and moreover

- (1) For each $v \in F^0$, we have $D_v = \{Y \cap F^1 \mid Y \in C_v, Y \cap F^1 \neq \emptyset\}$.
- (2) $T = \{Y \in S \mid Y \cap F^1 \neq \emptyset\}$.

Note that if (F, D, T) is a complete subobject of (E, C, S) , the inclusion $F \rightarrow E$ induces a morphism $(F, D, T) \rightarrow (E, C, S)$ in **SSGr**.

Condition (2) of the definition says that if $Y \in S$ and $Y \cap F^1 \neq \emptyset$, then $Y \subseteq F^1$. It follows that $T = S \cap D$. Note, however, that in the presence of condition (1), the equality $T = S \cap D$ is weaker than condition (2). This definition generalizes the usual definition of complete subgraphs of row-finite graphs, given in [9, p. 161], if we interpret a row-finite graph E as a triple (E, C, S) with $C_v = \{s^{-1}(v)\}$ if $s^{-1}(v) \neq \emptyset$ and $C_v = \emptyset$ if $s^{-1}(v) = \emptyset$, and $S = C_{\text{fin}} = C$.

Proposition 3.5. *Every object (E, C, S) in **SSGr** is the direct limit of its finite complete subobjects (F, D, T) , that is, of its complete subobjects (F, D, T) such that both F^0 and F^1 are finite sets.*

Proof. To show that an object (E, C, S) of **SSGr** is the direct limit of its finite complete subobjects, it is enough to prove that for every finite subset A of $E^0 \sqcup E^1$, there is a finite complete subobject (F, D, T) of (E, C, S) such that $A \subseteq F^0 \sqcup F^1$. Let E_1 be the subgraph of E generated by A , that is, $E_1^1 = A \cap E^1$ and $E_1^0 = (A \cap E^0) \cup s_E(E_1^1) \cup r_E(E_1^1)$. For $v \in E_1^0$, set

$$F_v := s_{E_1}^{-1}(v) \cup \bigcup_{X \in S \cap C_v, X \cap A \neq \emptyset} X.$$

Observe that F_v is a finite set for every $v \in E_1^0$. Let F be the subgraph of E generated by $E_1^0 \sqcup \bigsqcup_{v \in E_1^0} F_v$, so that $s_F^{-1}(v) = F_v$ for $v \in E_1^0 \subseteq F^0$ and $s_F^{-1}(v) = \emptyset$ for $v \in F^0 \setminus E_1^0$. For $v \in F^0$, set

$$D_v := \{Y \cap F^1 \mid Y \in C_v, Y \cap F^1 \neq \emptyset\} = \{Y \cap F^1 \mid Y \in C_v, Y \cap A \neq \emptyset\},$$

and then set $D := \bigsqcup_{v \in F^0} D_v$. Finally, set

$$T := \{Y \in S \mid Y \cap F^1 \neq \emptyset\} = S \cap D.$$

It follows that (F, D, T) is a finite complete subobject of (E, C, S) such that $A \subseteq F^0 \sqcup F^1$, as desired. \square

Recall that a functor is said to be *continuous* if it preserves direct limits.

Proposition 3.6. *The assignment $(E, C, S) \mapsto CL_K(E, C, S)$ extends to a continuous functor CL_K from the category **SSGr** to the category of (not necessarily unital) K -algebras. Moreover, every algebra $CL_K(E, C, S)$ is the direct limit, with injective transition maps, of the algebras $CL_K(F, D, T)$, where (F, D, T) runs over the directed system of all the finite complete subobjects of (E, C, S) .*

Proof. Let $\phi: (E, C, S) \rightarrow (E', C', S')$ be a morphism in **SSGr**. We check that there is a unique K -algebra homomorphism $CL_K(\phi): CL_K(E, C, S) \rightarrow CL_K(E', C', S')$ such that

$$CL_K(\phi)(v) = \phi^0(v), \quad CL_K(\phi)(e) = \phi^1(e), \quad CL_K(\phi)(e^*) = \phi^1(e)^*,$$

for $v \in E^0$ and $e \in E^1$. The relations (V) are preserved because ϕ^0 is injective, while (E1) and (E2) are preserved because ϕ is a graph homomorphism. Now, let us show that (SCK1), and (SCK2) for the members of S , are preserved by $CL_K(\phi)$. Let $v \in E^0$ and $X \in C_v$. Then there is (a unique) $Y \in C'_{\phi^0(v)}$ such that ϕ^1 restricts to an injective map $X \rightarrow Y$. Then for $e, f \in X$ we have $\phi^1(e), \phi^1(f) \in Y$ and so $\phi^1(e)^* \phi^1(f) = \delta_{\phi^1(e), \phi^1(f)} r(\phi^1(e)) = \delta_{e, f} \phi^0(r(e)) = CL_K(\phi)(e^* f)$. This verifies (SCK1). Note that we need ϕ^1 to be injective on X in order to guarantee that $\delta_{\phi^1(e), \phi^1(f)} = \delta_{e, f}$. Now assume that the above set X belongs to S . Then ϕ^1 restricts to a bijection $X \rightarrow Y$, so that

$$\phi^0(v) = \sum_{e \in X} \phi^1(e) \phi^1(e)^*,$$

showing that the relation (SCK2) for $X \in S$ is preserved by $CL_K(\phi)$. Since the defining relations for $CL_K(E, C, S)$ are preserved by $CL_K(\phi)$, the existence and uniqueness of this map follows. Functoriality is clear.

If (F, D, T) is a complete subobject of (E, C, S) and $\phi: (F, D, T) \rightarrow (E, C, S)$ is the inclusion, then the homomorphism $CL_K(\phi): CL_K(F, D, T) \rightarrow CL_K(E, C, S)$ is injective. This follows from Theorem 2.7. Indeed, if a monomial from $CL_K(F, D, T)$ as in (2.1) is in reduced form with respect to T , then this monomial will be also in reduced form with respect to S , thanks to properties (1) and (2) of the definition of complete subobject, assuming that we have made a coherent choice of edges $e_X \in X$ for $X \in T = \{Y \in S \mid Y \cap F^1 \neq \emptyset\}$. It

follows from Theorem 2.7 that $CL_K(\phi)$ maps a basis for $CL_K(F, D, T)$ to a subset of a basis for $CL_K(E, C, S)$, and therefore $CL_K(\phi)$ is injective.

We show now that the functor CL_K is continuous. Consider a directed system in the category **SSGr**, say $\{(E_i, C_i, S_i), \phi_{ji} \mid i, j \in I, j \geq i\}$, with direct limit (E, C, S) and limit maps $\eta_i : (E_i, C_i, S_i) \rightarrow (E, C, S)$. Let the K -algebra A be the direct limit of the directed system $\{CL_K(E_i, C_i, S_i), CL_K(\phi_i)\}$, with limit maps $\lambda_i : CL_K(E_i, C_i, S_i) \rightarrow A$. There is a unique K -algebra homomorphism $\theta : A \rightarrow CL_K(E, C, S)$ such that $\theta\lambda_i = CL_K(\eta_i)$ for all i , and we must show that θ is an isomorphism. Since each $CL_K(E_i, C_i, S_i)$ is generated by $E_i^0 \sqcup \hat{E}_i^1$, we see that A is generated by $\bigcup_{i \in I} \lambda_i(E_i^0 \sqcup \hat{E}_i^1)$. Given any $w \in E^0$, write $w = \eta_i^0(v)$ for some $i \in I$ and $v \in E_i^0$, and set $\xi^0(w) = \lambda_i(v) \in A$. Note that $\xi^0(w)$ is independent of the representation $w = \eta_i^0(v)$, and so we have a well-defined map $\xi^0 : E^0 \rightarrow A$. Similarly, there is a well-defined map $\xi^1 : \hat{E}^1 \rightarrow A$ such that $\xi^1(\eta_i^1(e)) = \lambda_i(e)$ and $\xi^1(\eta_i^1(e)^*) = \lambda_i(e)^*$ for all $i \in I$ and $e \in E_i^1$. Observe that the elements $\xi^0(w)$, for $w \in E^0$, and $\xi^1(e)$, $\xi^1(e)^*$, for $e \in E^1$, satisfy the defining relations of $CL_K(E, C, S)$. Hence, $\xi^0 \sqcup \xi^1$ extends uniquely to a K -algebra homomorphism $\xi : CL_K(E, C, S) \rightarrow A$, and ξ is an inverse for θ . Therefore θ is an isomorphism, as required.

Finally, observe that, by Proposition 3.5, any object (E, C, S) in **SSGr** is the direct limit in **SSGr** of the directed system of its finite complete subobjects. This, together with what we have already shown before, gives the desired result about the direct limit representation of $CL_K(E, C, S)$. \square

With another dose of direct limits, we can prove that all Cohn-Leavitt algebras are hereditary in a suitable non-unital sense (see Definition 10.3). This relies on work of Bergman and Dicks [15], which requires us to work with direct limits of unital algebras over intervals of ordinals. Let us write A^\sim for the K -algebra unitization of any K -algebra A , as in Definition 10.5.

The following easy observation will be useful: If $(R_t)_{t \in T}$ is a nonempty family of unital hereditary K -algebras, then $R := (\bigoplus_{t \in T} R_t)^\sim$ is hereditary. This holds because every left ideal of R has one of the forms $\bigoplus_t I_t$ or $R(1 - e) \oplus \bigoplus_{j=1}^n I_{t_j}$ where I_t is a left ideal of R_t and e (in the second case) is the sum of the identity elements of R_{t_1}, \dots, R_{t_n} , and similarly for right ideals.

Theorem 3.7. *For any object (E, C, S) of **SSGr**, the algebra $CL_K(E, C, S)$ is hereditary.*

Proof. Set $A := CL_K(E, C, S)$. By Proposition 10.7, it suffices to show that A^\sim is hereditary.

We first construct an algebra that corresponds to the unitization of the subalgebra of A generated by E^0 together with the idempotents ee^* for edges e lying in sets $X \in C \setminus S$. There are three steps:

- (1) For each $X \in C \setminus S$, let B_X be the K -algebra direct sum of $|X|$ copies of K .
- (2) For each $v \in E^0$, let A_v be the unital K -algebra coproduct of the algebras B_X^\sim for $X \in C_v \setminus S$. (The coproduct of an empty family is K itself.)
- (3) Set $A_0 := (\bigoplus_{v \in E^0} A_v)^\sim$.

Note that each B_X has a K -basis $(b_e)_{e \in X}$ consisting of a family of $|X|$ pairwise orthogonal idempotents. By the observation above, B_X^\sim is a unital (commutative) hereditary K -algebra. Next, [15, Theorem 3.4] implies that each A_v is hereditary, and then A_0 is hereditary by the observation above. For $v \in E^0$, identify v with the identity element of A_v . Then set $E_0 = (E^0, \emptyset)$, and identify $L_K(E_0)$ with the subalgebra $\bigoplus_{v \in E^0} Kv \subseteq A_0$. There is a unique unital K -algebra homomorphism $\phi_0 : A_0 \rightarrow A^\sim$ such that $\phi_0(v) = v$ for $v \in E^0$ and $\phi_0(b_e) = ee^*$ for $e \in X \in C \setminus S$.

Now let E_1^1 be the union of all the sets $X \in C \setminus S$. Set $E_1 = (E^0, E_1^1)$ and $C_1 = C \setminus S$. We next construct an algebra corresponding to $C_K(E_1, C_1)^\sim$. For the purpose, choose an ordinal γ and a bijection $\alpha \mapsto e_\alpha$ from $[0, \gamma)$ to E_1^1 . We build unital K -algebras A_α for $\alpha \in [0, \gamma]$ as follows. (We do not specifically label the obvious connecting homomorphisms $A_\alpha \rightarrow A_\beta$ for $0 \leq \alpha < \beta \leq \gamma$ that the construction carries, and we treat them as inclusion maps.)

- (4) Start with A_0 as in (3).
- (5) For $\alpha \in [0, \gamma)$, let $A_{\alpha+1}$ be obtained from A_α by adjoining a universal isomorphism between the finitely generated projective left A_α -modules $A_\alpha b_{e_\alpha}$ and $A_\alpha r(e_\alpha)$.
- (6) If $\beta \leq \gamma$ is a limit ordinal and A_α has been defined for all $\alpha < \beta$, take A_β to be the direct limit of $(A_\alpha)_{\alpha < \beta}$.

We apply [15, Theorem 3.4] a second time to see that A_γ is hereditary. For $\alpha \in [0, \gamma)$, the algebra $A_{\alpha+1}$ is generated by A_α together with elements x_α and y_α universally satisfying the relations

$$x_\alpha = b_{e_\alpha} x_\alpha r(e_\alpha) \quad y_\alpha = r(e_\alpha) y_\alpha b_{e_\alpha} \quad x_\alpha y_\alpha = b_{e_\alpha} \quad y_\alpha x_\alpha = r(e_\alpha).$$

The homomorphism ϕ_0 extends uniquely to a unital K -algebra homomorphism $\phi_1 : A_\gamma \rightarrow A^\sim$ such that $\phi_1(x_\alpha) = e_\alpha$ and $\phi_1(y_\alpha) = e_\alpha^*$ for $\alpha \in [0, \gamma)$.

Finally, we construct a further direct limit to reach A^\sim . Choose an ordinal $\mu > \gamma$ and a bijection $\kappa \mapsto X_\kappa$ from $[\gamma, \mu)$ to S . Then build unital K -algebras A_κ for $\kappa \in [\gamma, \mu]$ as follows.

- (7) Start with A_γ as in the previous construction.
- (8) For $\kappa \in [\gamma, \mu)$, let $A_{\kappa+1}$ be obtained from A_κ by adjoining a universal isomorphism between the finitely generated projective A_κ -modules $A_\kappa v_\kappa$ and $\bigoplus_{e \in X_\kappa} A_\kappa r(e)$, where $v_\kappa \in E^0$ is such that $X_\kappa \in C_{v_\kappa} \cap S$.
- (9) If $\lambda \in (\gamma, \mu]$ is a limit ordinal and A_κ has been defined for all $\kappa < \lambda$, take A_λ to be the direct limit of $(A_\kappa)_{\kappa < \lambda}$.

A third application of [15, Theorem 3.4] yields that A_μ is hereditary. For $\kappa \in [\gamma, \mu)$, the algebra $A_{\kappa+1}$ is generated by A_κ together with elements x_e and y_e for $e \in X_\kappa$ universally satisfying the relations

$$\begin{aligned} x_e &= v_\kappa x_e r(e) & y_e &= r(e) y_e v_\kappa \\ \sum_{e \in X_\kappa} x_e y_e &= v_\kappa & y_e x_f &= \delta_{e,f} r(e) \end{aligned}$$

for $e, f \in X_\kappa$. The homomorphism ϕ_1 extends uniquely to a unital K -algebra homomorphism $\phi : A_\mu \rightarrow A^\sim$ such that $\phi(x_e) = e$ and $\phi(y_e) = e^*$ for $e \in X_\kappa$, $\kappa \in [\gamma, \mu)$.

In the reverse direction, there is a unique unital K -algebra homomorphism $\psi : A^\sim \rightarrow A_\mu$ such that

$$\begin{aligned} \psi(v) &= v & (v \in E^0) \\ \psi(e_\alpha) &= x_\alpha \text{ and } \psi(e_\alpha^*) = y_\alpha & (\alpha \in [0, \gamma)) \\ \psi(e) &= x_e \text{ and } \psi(e^*) = y_e & (e \in X_\kappa, \kappa \in [\gamma, \mu)). \end{aligned}$$

Clearly, ψ and ϕ are mutual inverses. Therefore $A_\mu \cong A^\sim$, and the theorem is proved. \square

4. THE MONOID $\mathcal{V}(CL_K(E, C, S))$

We define an abelian monoid $M(E, C, S)$ for any separated graph (E, C) with a distinguished subset $S \subseteq C_{\text{fin}}$, and we prove that $\mathcal{V}(CL_K(E, C, S))$ is naturally isomorphic to $M(E, C, S)$. This extends previous results for Leavitt path algebras $L_K(E)$ of row-finite graphs E [9, Theorem 3.5], for which case it was also proved that the graph monoid $M(E) \cong \mathcal{V}(L_K(E))$ is a refinement monoid [9, Proposition 4.4]. We shall prove that $M(E)$ is a refinement monoid for any non-separated graph E (Corollary 5.16). In contrast, refinement does not always hold in $\mathcal{V}(CL_K(E, C, S))$ – in fact, $\mathcal{V}(CL_K(E, C, S))$ can be an arbitrary conical monoid (Proposition 4.4).

Definition 4.1. Let (E, C, S) be an object in **SSGr**. We define the *graph monoid* $M(E, C, S)$ as the abelian monoid given by the set of generators

$$E^0 \sqcup \{q'_Z \mid Z \subseteq X \in C, 0 < |Z| < \infty\}$$

and the following relations:

- (1) $v = \mathbf{r}(Z) + q'_Z$ for $v \in E^0$, $Z \subseteq X \in C_v$, and $0 < |Z| < \infty$, where for a finite subset Y of E^1 we set $\mathbf{r}(Y) := \sum_{e \in Y} r(e)$.
- (2) $q'_{Z_1} = \mathbf{r}(Z_2 \setminus Z_1) + q'_{Z_2}$ for finite nonempty subsets Z_1 and Z_2 of $X \in C$ with $Z_1 \subsetneq Z_2$.
- (3) $q'_X = 0$ for $X \in S$.

Of course the elements q'_Z are intended to represent the equivalence classes of the idempotents $v - \sum_{e \in Z} ee^*$ in $CL_K(E, C, S)$, for Z a finite nonempty subset of $X \in C_v$.

There is some redundancy among these generators and relations. In particular, we could omit the generators q'_Z for nonempty proper subsets Z of a set $X \in C_{\text{fin}}$, since relation (2) gives q'_Z in terms of q'_X , and relation (1) for Z follows from the corresponding relation for X in light of (2). In general, (1) could be viewed as a form of (2) with $Z_1 = \emptyset$, except that the notation q'_\emptyset would not be well-defined.

When working with objects (E, C) from **SGr**, we abbreviate the notation for the corresponding monoid to

$$M(E, C) := M(E, C, C_{\text{fin}}).$$

For many later purposes, we shall assume that (E, C) is finitely separated. In that case, the generators q'_Z for $\emptyset \neq Z \subsetneq X \in C$ are redundant, as noted above. Then, we can present $M(E, C, S)$ with the set of generators $E^0 \sqcup \{q'_X \mid X \in C \setminus S\}$ and the relations

- (4) $v = \mathbf{r}(X)$ for $v \in E^0$ and $X \in C_v \cap S$.

(5) $v = \mathbf{r}(X) + q'_X$ for $v \in E^0$ and $X \in C_v \setminus S$.

Now return to the general case, and consider a morphism $\phi : (E, C, S) \rightarrow (E', C', S')$ in **SSGr**. There is a unique monoid homomorphism $M(\phi) : M(E, C, S) \rightarrow M(E', C', S')$ sending $v \mapsto \phi^0(v)$ for $v \in E^0$ and $q'_Z \mapsto q'_{\phi^1(Z)}$ for nonempty finite sets $Z \subseteq X \in C$. (The latter assignments are well-defined because if Z is a nonempty finite subset of some $X \in C$, then $\phi^1(Z)$ is a nonempty finite subset of $\phi(X) \in C'$.) The assignments $(E, C, S) \mapsto M(E, C, S)$ and $\phi \mapsto M(\phi)$ define a functor M from **SSGr** to the category **Mon** of abelian monoids. It is easily checked (just as for the functor CL_K in Proposition 3.6) that M is continuous.

Lemma 4.2. *If (E, C, S) is an object in **SSGr**, then $M(E, C, S)$ is a nonzero, conical monoid.*

Proof. We can present $M(E, C, S)$ as the quotient of the free abelian monoid F on the set

$$E^0 \sqcup \{q'_Z \mid Z \subseteq X \in C, 0 < |Z| < \infty\} \setminus \{q'_X \mid X \in S\}$$

modulo the congruence \sim generated by the relations (1) and (2), where (1) is rewritten $v = \mathbf{r}(Z)$ in case $Z = X \in S$ and (2) is rewritten $q'_{Z_1} = \mathbf{r}(Z_2 \setminus Z_1)$ in case $Z_2 = X \in S$. Observe that $\alpha \sim 0$ occurs in F only when $\alpha = 0$. The lemma follows immediately, given that E^0 is assumed to be nonempty. \square

Theorem 4.3. *There is an isomorphism $\Gamma : M \rightarrow \mathcal{V} \circ CL_K$ of functors $\mathbf{SSGr} \rightarrow \mathbf{Mon}$, given as follows. For each object (E, C, S) of **SSGr**,*

$$\Gamma(E, C, S) : M(E, C, S) \rightarrow \mathcal{V}(CL_K(E, C, S))$$

is the monoid homomorphism sending $v \mapsto [v]$ for $v \in E^0$ and $q'_Z \mapsto [v - \sum_{e \in Z} ee^]$ for finite nonempty subsets $Z \subseteq X \in C_v$.*

Proof. It is easily seen that the maps $\Gamma(E, C, S)$ are well-defined monoid homomorphisms, and that Γ defines a natural transformation from M to $\mathcal{V} \circ CL_K$.

We have observed that M is continuous, and so is $\mathcal{V} \circ CL_K$, taking into account that \mathcal{V} is continuous and Proposition 3.6. Thus, by making use of the second part of Proposition 3.6 we see that it is sufficient to show that $\Gamma(E, C, S)$ is an isomorphism in the case where E is a finite graph.

We use induction on $|C|$ to establish the result for finite objects (E, C, S) in **SSGr**. The result is trivial if $|C| = 0$ (i.e., if there are no edges in E). Assume that $\Gamma(E, C, S)$ is an isomorphism for finite objects (F, D, T) of **SSGr** with $|D| < n - 1$ for some $n \geq 1$, and let (E, C, S) be a finite object in **SSGr** such that $|C| = n$. Select X in C_v , for some $v \in E^0$. We can apply induction to the triple (F, D, T) obtained from (E, C, S) by deleting all the edges in X , and leaving intact the structure corresponding to the remaining subsets $Y \in C$ (keeping $F^0 = E^0$).

Assume first that $X \in S$. Then $M(E, C, S)$ is obtained from $M(F, D, T)$ by factoring out the relation $v = \mathbf{r}(X)$. On the other hand, the algebra $CL_K(E, C, S)$ is the Bergman algebra obtained from $CL_K(F, D, T)$ by adjoining a universal isomorphism between the finitely generated projective modules $CL_K(F, D, T)v$ and $\bigoplus_{e \in X} CL_K(F, D, T)r(e)$. Accordingly, it

follows from [13, Theorem 5.2] that $\mathcal{V}(CL_K(E, C, S))$ is the quotient of $\mathcal{V}(CL_K(F, D, T))$ modulo the relation $[v] = [\mathbf{r}(X)]$. Since $\Gamma(F, D, T) : M(F, D, T) \rightarrow \mathcal{V}(CL_K(F, D, T))$ is an isomorphism by the induction hypothesis, we obtain that $\Gamma(E, C, S)$ is an isomorphism in this case.

Assume now that $X \notin S$. In this case, $M(E, C, S)$ is obtained from $M(F, D, T)$ by adjoining a new generator q'_X and factoring out the relation $v = \mathbf{r}(X) + q'_X$. On the K -algebra side, we shall make use of another of Bergman's constructions, namely "the creation of idempotents". Write $X = \{e_1, \dots, e_m\}$, and recall that $X \in C_v$, so that $s_E(e_i) = v$ for all i . Let R be the algebra obtained from $CL_K(F, D, T)$ by adjoining $m + 1$ pairwise orthogonal idempotents g_1, \dots, g_m, q_X with

$$v = g_1 + \dots + g_m + q_X.$$

It follows from [13, Theorem 5.1] that $\mathcal{V}(R)$ is the monoid obtained from $\mathcal{V}(CL_K(F, D, T))$ by adjoining $m+1$ new generators z_1, \dots, z_m, q''_X and factoring out the relation $[v] = \sum_{j=1}^m z_j + q''_X$.

Now, it is clear that $CL_K(E, C, S)$ is isomorphic to the Bergman algebra obtained from R by consecutively adjoining universal isomorphisms between the left modules generated by the idempotents $r(e_i)$ and g_i , for $i = 1, \dots, m$. It follows that $\mathcal{V}(CL_K(E, C, S))$ is the monoid obtained from $\mathcal{V}(CL_K(F, D, T))$ by adjoining a new generator q''_X and factoring out the relation $[v] = [\mathbf{r}(X)] + q''_X$. Therefore, applying the induction hypothesis to (F, D, T) , we again conclude that $\Gamma(E, C, S)$ is an isomorphism. \square

We conclude this section by noting that all conical abelian monoids appear as graph monoids of separated graphs.

Proposition 4.4. *If M is any conical abelian monoid, there exists a finitely separated graph (E, C) such that*

$$M \cong M(E, C) \cong \mathcal{V}(L_K(E, C)).$$

Proof. Choose a presentation of M with a nonempty set $\{x_j \mid j \in J\}$ of generators and a nonempty set $\{r_i \mid i \in I\}$ of relations

$$(4.1) \quad r_i : \quad \sum_{j \in J} a_{ij} x_j = \sum_{j \in J} b_{ij} x_j,$$

where for each i , we have $a_{ij} \neq 0$ for at least one but only finitely many $j \in J$, and similarly for the b_{ij} . The relations can be chosen with these restrictions because M is conical. For instance, if, for some i , all $a_{ij} = 0$, then $\sum_j b_{ij} x_j = 0$ and consequently $x_j = 0$ whenever $b_{ij} > 0$. In this case, we may delete those x_j for which $b_{ij} > 0$ from our set of generators.

Let (E, C) be the finitely separated graph constructed as follows:

- (1) $E^0 := \{u_i \mid i \in I\} \sqcup \{v_j \mid j \in J\}$.
- (2) All the u_i are sources, and all the v_j are sinks.
- (3) For each $i \in I$ and $j \in J$, there are exactly $a_{ij} + b_{ij}$ edges with source u_i and range v_j .
- (4) Each $s^{-1}(u_i) = X_{i1} \sqcup X_{i2}$ where X_{i1} contains exactly a_{ij} edges $u_i \rightarrow v_j$ for each j , and X_{i2} contains exactly b_{ij} edges $u_i \rightarrow v_j$ for each j . Thus, $E^1 = \bigsqcup_{i \in I} (X_{i1} \sqcup X_{i2})$.
- (5) $C := \bigsqcup_{v \in E^0} C_v$ where each $C_{u_i} := \{X_{i1}, X_{i2}\}$ and each $C_{v_j} := \emptyset$.

Since (E, C) is finitely separated, the monoid $M(E, C)$ can be presented (as noted in Definition 4.1) with the set of generators E^0 and the relations $v = \mathbf{r}(X)$ for $v \in E^0$ and $X \in C_v$. These relations are of the form

$$u_i = \sum_{j \in J} a_{ij} v_j \quad \text{and} \quad u_i = \sum_{j \in J} b_{ij} v_j$$

for $i \in I$. The isomorphism $M \cong M(E, C)$ follows immediately, and the final isomorphism is given by Theorem 4.3. \square

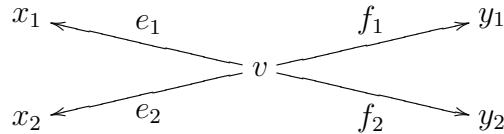
Corollary 4.5. *Let M be a conical abelian monoid. Then there exists a hereditary K -algebra $A := L_K(E, C)$, for some finitely separated graph (E, C) , such that $\mathcal{V}(A) \cong M$.*

Proof. Theorems 3.7 and 4.3 and Proposition 4.4. \square

Corollary 4.5 incorporates a more explicit rendering of a result of Bergman [13, Theorem 6.4], as corrected and extended by Bergman and Dicks [15, Remarks following Theorem 3.4]: Any conical abelian monoid with an order-unit is isomorphic to $\mathcal{V}(R)$ for some unital hereditary K -algebra R . Our development does not replace the mentioned result, since the theorems proved in [13] are crucial to our proof of Theorem 4.3, and our algebras are mostly non-unital.

5. REFINEMENT

In this section we initiate our systematic study of the properties of the graph monoids $M(E, C, S)$ associated to separated graphs (E, C) with distinguished subsets $S \subseteq C_{\text{fin}}$, focusing in particular on the Riesz refinement property. For non-separated, row-finite graphs E , the corresponding monoid $M(E)$ was proved to have refinement in [9, Proposition 4.4]. This result can be extended to general graphs E , once the definition of $M(E)$ is modified along the lines of Definition 4.1 (use Theorem 4.3 and [26, Theorem 5.8]). It also follows from our results here (see Corollary 5.16). However, refinement does not always hold in $M(E, C, S)$ or even in $M(E, C)$. For example, let E be the graph



and take $C_v = \{X, Y\}$ where $X = \{e_1, e_2\}$ and $Y = \{f_1, f_2\}$. Then $M(E, C)$ is the monoid presented by generators x_1, x_2, y_1, y_2 and the relation $x_1 + x_2 = y_1 + y_2$. This is not a refinement monoid. Thus, extra conditions on (E, C, S) are needed to obtain refinement in $M(E, C, S)$.

We obtain refinement by imposing conditions of the following type. Suppose we have an object $(E, C, S) \in \mathbf{SSGr}$, a vertex $v \in E^0$, and distinct sets $X, Y \in S \cap C_v$. Then $v = \mathbf{r}(X) = \mathbf{r}(Y)$ in $M(E, C, S)$, and we require assumptions that allow a refinement of the equation $\mathbf{r}(X) = \mathbf{r}(Y)$. A sufficient condition is the existence of edges $g_{e,f}, h_{e,f} \in E^1$ for $e \in X$ and $f \in Y$, with $s(g_{e,f}) = r(e)$, $s(h_{e,f}) = r(f)$, and $r(g_{e,f}) = r(h_{e,f})$ for all e, f ,

such that the sets $X_e := \{g_{e,f} \mid f \in Y\}$ and $Y_f := \{h_{e,f} \mid e \in X\}$ belong to $C_{r(e)} \cap S$ and $C_{r(f)} \cap S$ respectively. Then, in $M(E, C, S)$, we would have $r(e) = \sum_{f \in Y} r(g_{e,f})$ for $e \in X$ and $r(f) = \sum_{e \in X} r(h_{e,f}) = \sum_{e \in X} r(g_{e,f})$ for $f \in Y$, providing the desired refinement of $\mathbf{r}(X) = \mathbf{r}(Y)$.

We actually impose a slightly weaker assumption, which we place on the free abelian monoid on a generating set for $M(E, C, S)$ (see Definition 5.12). This assumption has a smoother form when all the sets in C are finite (Definition 5.2), and so we begin with that case. An easy reduction will allow us to assume, in addition, that $S = C$. Later, we reduce the general case to the one just mentioned.

Assumption. Throughout this section, (E, C, S) will denote a fixed, arbitrary object of **SSGr**. Until Proposition 5.9, we also assume that (E, C) is a finitely separated graph (that is, $C = C_{\text{fin}}$).

Set $Q^0 := \{q'_X \mid X \in C \setminus S\}$, and let F be the free abelian monoid on $E^0 \sqcup Q^0$. (We use the same notation for elements of $E^0 \sqcup Q^0$ in F as for their images in $M(E, C, S)$.) As above, set $\mathbf{r}(X) = \sum_{e \in X} r(e)$ for $X \in C$ (in either F or $M(E, C, S)$). Then set

$$\rho(X) = \rho_E(X) := \begin{cases} \mathbf{r}(X) & (X \in S) \\ \mathbf{r}(X) + q'_X & (X \in C \setminus S). \end{cases}$$

We identify $M(E, C, S)$ with the quotient monoid F/\sim , where \sim is the congruence on F generated by the relations

$$v \sim \rho(X)$$

for $v \in E^0$ and $X \in C_v$.

Definition 5.1. [Assuming $C = C_{\text{fin}}$] For $\alpha, \beta \in F$, write $\alpha \rightsquigarrow_1 \beta$ to denote the following situation:

$$\alpha = \sum_{i=1}^k u_i + \sum_{i=k+1}^m u_i \text{ and } \beta = \sum_{i=1}^k \rho(X_i) + \sum_{i=k+1}^m u_i \text{ for some } u_i \in E^0 \sqcup Q^0 \text{ and some } X_i \in C_{u_i}, \text{ where } u_1, \dots, u_k \in E^0 \setminus \text{Sink}(E).$$

We view $\alpha = 0$ as an empty sum of the above form (i.e., $k = m = 0$), so that $0 \rightsquigarrow_1 0$. Note also, taking $k = 0$, that $\alpha \rightsquigarrow_1 \alpha$ for all $\alpha \in F$.

Now write $\alpha \rightsquigarrow_n \beta$, where $n \geq 1$, in case there is a finite string $\alpha = \alpha_0 \rightsquigarrow_1 \alpha_1 \rightsquigarrow_1 \dots \rightsquigarrow_1 \alpha_n = \beta$. Finally, $\alpha \rightsquigarrow \beta$ just means that $\alpha \rightsquigarrow_n \beta$ for some n . Observe that if $\alpha \rightsquigarrow_m \beta$ and $\alpha' \rightsquigarrow_n \beta'$, then $\alpha + \alpha' \rightsquigarrow_s \beta + \beta'$ with $s = \max\{m, n\}$. (This follows from the fact that $\alpha \rightsquigarrow_1 \beta$ implies $\alpha + \gamma \rightsquigarrow_1 \beta + \gamma$ for all $\gamma \in F$.)

Definition 5.2. [Assuming $C = C_{\text{fin}}$] **Assumption (*):** For all $v \in E^0$ and all $X, Y \in C_v$, there exists $\gamma \in F$ such that $\rho(X) \rightsquigarrow_1 \gamma$ and $\rho(Y) \rightsquigarrow_1 \gamma$.

This assumption only needs to be imposed when X and Y are disjoint, since otherwise $X = Y$ and the conclusion is trivially satisfied. Upcoming inductions (see the proof of Lemma 5.7) require the use of \rightsquigarrow_1 rather than \rightsquigarrow in Assumption (*).

Construction 5.3. Enlarge E to a graph \tilde{E} by adjoining new vertices w_X for $X \in C \setminus S$ and new edges $f_X : v \rightarrow w_X$ for $v \in E^0$ and $X \in C_v \setminus S$. For $v \in E^0$ and $X \in C_v$, set

$$\tilde{X} := \begin{cases} X & (X \in S) \\ X \sqcup \{f_X\} & (X \notin S), \end{cases}$$

and then set $\tilde{C}_v = \{\tilde{X} \mid X \in C_v\}$, a partition of $s_{\tilde{E}}^{-1}(v)$. Since the vertices $w_X \in \tilde{E}^0$ are sinks, we just set $\tilde{C}_{w_X} = \emptyset$ for $X \in C \setminus S$. We have now built a separated graph (\tilde{E}, \tilde{C}) , where $\tilde{C} = \bigsqcup_{w \in \tilde{E}} \tilde{C}_w$. Since all the sets in C are finite, so are those in \tilde{C} .

Take $\tilde{S} = \tilde{C}$, and observe that $M(\tilde{E}, \tilde{C}) = M(\tilde{E}, \tilde{C}, \tilde{S})$ is the abelian monoid presented by the generating set \tilde{E}^0 and the relations $w = \mathbf{r}_{\tilde{E}}(\tilde{X})$ for $w \in \tilde{E}^0$ and $\tilde{X} \in \tilde{C}_w$. These relations are of two types:

$$\begin{aligned} v &= \mathbf{r}_E(X) & (v \in E^0, X \in C_v \cap S) \\ v &= \mathbf{r}_E(X) + w_X & (v \in E^0, X \in C_v \setminus S); \end{aligned}$$

there are no relations for $w \in \tilde{E}^0 \setminus E^0$. Comparing presentations, we see that there is a monoid isomorphism

$$M(E, C, S) \longrightarrow M(\tilde{E}, \tilde{C})$$

sending $v \mapsto v$ for $v \in E^0$ and $q'_X \mapsto w_X$ for $X \in C \setminus S$.

Now let \tilde{F} be the free abelian monoid on \tilde{E}^0 , and let $\theta : F \rightarrow \tilde{F}$ be the isomorphism such that $\theta(v) = v$ for $v \in E^0$ and $\theta(q'_X) = w_X$ for $X \in C \setminus S$. Observe that

$$\theta(\boldsymbol{\rho}_E(X)) = \mathbf{r}_{\tilde{E}}(\tilde{X})$$

for all $X \in C$. It follows that if $\alpha, \beta \in F$ are any elements satisfying $\alpha \rightsquigarrow_1 \beta$, then $\theta(\alpha) \rightsquigarrow_1 \theta(\beta)$ in \tilde{F} . It is now clear that if Assumption $(*)$ holds for (E, C, S) , then it also holds for (\tilde{E}, \tilde{C}) .

We aim to prove that when $(*)$ holds, $M(E, C, S)$ is a refinement monoid. By what we have just shown, it suffices to prove this result for $M(\tilde{E}, \tilde{C})$. Thus:

Assumption. Until Proposition 5.9, we assume that $S = C$.

With the above simplification, F is the free abelian monoid on E^0 , and $\boldsymbol{\rho}(X) = \mathbf{r}(X)$ for $X \in C$.

We will need the following relation \rightarrow_1 , a more elementary version of \rightsquigarrow_1 , which provides an alternative way to build \rightsquigarrow .

Definition 5.4. For nonzero $\alpha, \beta \in F$, write $\alpha \rightarrow_1 \beta$ to denote the following situation:

$$\alpha = \sum_{i=1}^n v_i \text{ and } \beta = \mathbf{r}(X) + \sum_{i=2}^n v_i \text{ for some } v_i \in E^0 \text{ and some } X \in C_{v_1}, \text{ where } v_1 \notin \text{Sink}(E).$$

Observe that \sim is the congruence on F generated by \rightarrow_1 .

Let \rightarrow be the transitive and reflexive closure of \rightarrow_1 on F , that is, $\alpha \rightarrow \beta$ if and only if there is a finite string $\alpha = \alpha_0 \rightarrow_1 \alpha_1 \rightarrow_1 \cdots \rightarrow_1 \alpha_t = \beta$, in which case we write $\alpha \rightarrow_t \beta$. In particular, $t = 0$ is allowed, so that $\alpha \rightarrow \alpha$ for all $\alpha \in F$ (including $\alpha = 0$).

Observe that $\alpha \rightarrow_1 \beta$ implies $\alpha \rightsquigarrow_1 \beta$, so that $\alpha \rightarrow_n \beta$ implies $\alpha \rightsquigarrow_n \beta$ for any $n > 0$. On the other hand, the basic instance of $\alpha \rightsquigarrow_1 \beta$ given in Definition 5.1 can clearly be achieved by k iterations of \rightarrow_1 . Consequently, $\alpha \rightsquigarrow_n \beta$ implies $\alpha \rightarrow_t \beta$ for some $t \geq n$. Therefore \rightarrow coincides with \rightsquigarrow .

Note that \rightarrow_1 is partially compatible with sums: $\alpha \rightarrow_1 \beta$ implies $\alpha + \gamma \rightarrow_1 \beta + \gamma$ for any $\gamma \in F$. Consequently, if $\alpha \rightarrow_s \beta$ and $\alpha' \rightarrow_t \beta'$, then $\alpha + \alpha' \rightarrow_{s+t} \beta + \beta'$. This of course yields that \rightarrow is compatible with sums: if $\alpha \rightarrow \beta$ and $\alpha' \rightarrow \beta'$, then $\alpha + \alpha' \rightarrow \beta + \beta'$.

Lemma 5.5. *Let $\alpha, \beta \in F$. Then $\alpha \sim \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n-1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$.*

The number n above will be called the *length* of the string. Strings of length zero are allowed.

Proof. Define a relation \approx on F by the given condition on strings. Namely, $\alpha \approx \beta$ if and only if there is a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n-1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. It is clear that \approx is an equivalence relation, and it follows from the partial compatibility of \rightarrow_1 with sums that \approx is a monoid congruence. As \rightarrow_1 implies \approx and \rightarrow_1 generates \sim , it follows that $\sim \subseteq \approx$. The reverse inclusion holds by construction of \approx . Therefore \approx coincides with \sim , proving the lemma. \square

The *support* of an element γ in F , denoted $\text{supp}(\gamma) \subseteq E^0$, is the set of basis elements appearing in the canonical expression of γ .

Lemma 5.6. *Assume that $\alpha, \alpha_1, \alpha_2, \beta \in F$ with $\alpha = \alpha_1 + \alpha_2$ and $\alpha \rightarrow_n \beta$ for some n . Then β can be written as $\beta = \beta_1 + \beta_2$ with $\alpha_1 \rightarrow_s \beta_1$ and $\alpha_2 \rightarrow_t \beta_2$, where $s, t \leq n$.*

Proof. By induction, it is enough to show the result in the case where $\alpha \rightarrow_1 \beta$. If $\alpha \rightarrow_1 \beta$, then there is an element v in the support of α such that $\beta = (\alpha - v) + \mathbf{r}(X)$ for some $X \in C_v$. The element v belongs either to the support of α_1 or to the support of α_2 . Assume, for instance, that v belongs to the support of α_1 . Then we set $\beta_1 = (\alpha_1 - v) + \mathbf{r}(X)$ and $\beta_2 = \alpha_2$. \square

Note that the elements β_1 and β_2 in Lemma 5.6 are not uniquely determined by α_1 and α_2 in general, because the element $v \in E^0$ considered in the proof could belong to *both* the support of α_1 and the support of α_2 .

Lemma 5.7. *Assume (*), and let $\alpha, \beta, \gamma \in F$. If $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$, there exists $\delta \in F$ such that $\beta \rightarrow \delta$ and $\gamma \rightarrow \delta$.*

Proof. Since \rightarrow coincides with \rightsquigarrow , it suffices to prove the corresponding confluence for \rightsquigarrow : If $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \gamma$, there exists $\delta \in F$ such that $\beta \rightsquigarrow \delta$ and $\gamma \rightsquigarrow \delta$.

If $\alpha = 0$, then $\beta = \gamma = 0$, and we take $\delta = 0$. Hence, we may assume that $\alpha \neq 0$, in which case $\beta, \gamma \neq 0$.

Claim 1: If $\alpha \rightsquigarrow_1 \beta$ and $\alpha \rightsquigarrow_1 \gamma$, there exists $\delta \in F$ such that $\beta \rightsquigarrow_1 \delta$ and $\gamma \rightsquigarrow_1 \delta$.

In this situation, we can write $\alpha = \sum_{i=1}^n u_i$ for some $u_i \in E^0$ and

$$\begin{aligned}\beta &= \sum_{i=1}^k \mathbf{r}(X_i) + \sum_{i=k+1}^l \mathbf{r}(X_i) + \sum_{i=l+1}^m u_i + \sum_{i=m+1}^n u_i \\ \gamma &= \sum_{i=1}^k \mathbf{r}(Y_i) + \sum_{i=k+1}^l u_i + \sum_{i=l+1}^m \mathbf{r}(Y_i) + \sum_{i=m+1}^n u_i\end{aligned}$$

with $0 \leq k \leq l \leq m \leq n$ and all $X_i, Y_i \in C_{u_i}$. By $(*)$, there exist $\delta_1, \dots, \delta_k \in F$ such that $\mathbf{r}(X_i) \rightsquigarrow_1 \delta_i$ and $\mathbf{r}(Y_i) \rightsquigarrow_1 \delta_i$ for all $i = 1, \dots, k$. Set

$$\delta = \sum_{i=1}^k \delta_i + \sum_{i=k+1}^l \mathbf{r}(X_i) + \sum_{i=l+1}^m \mathbf{r}(Y_i) + \sum_{i=m+1}^n u_i.$$

Then $\beta \rightsquigarrow_1 \delta$ and $\gamma \rightsquigarrow_1 \delta$. This verifies Claim 1.

Claim 2: If $\alpha \rightsquigarrow_1 \beta$ and $\alpha \rightsquigarrow_n \gamma$, there exists $\delta \in F$ such that $\beta \rightsquigarrow \delta$ and $\gamma \rightsquigarrow \delta$.

If $n = 1$, the claim follows from Claim 1. Now assume that $n > 1$, and write $\alpha \rightsquigarrow_1 \alpha' \rightsquigarrow_{n-1} \gamma$, for some $\alpha' \in F$. By Claim 1, there is some $\delta' \in F$ such that $\beta \rightsquigarrow_1 \delta'$ and $\alpha' \rightsquigarrow_1 \delta'$. By induction on n , there exists $\delta \in F$ such that $\delta' \rightsquigarrow \delta$ and $\gamma \rightsquigarrow \delta$. Since then $\beta \rightsquigarrow \delta$, Claim 2 is proved.

Claim 3: If $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow_n \gamma$, there exists $\delta \in F$ such that $\beta \rightsquigarrow \delta$ and $\gamma \rightsquigarrow \delta$.

The case $n = 1$ holds by Claim 2. Now assume that $n > 1$, and write $\alpha \rightsquigarrow_1 \alpha' \rightsquigarrow_{n-1} \gamma$, for some $\alpha' \in F$. By Claim 2, there is some $\delta' \in F$ such that $\beta \rightsquigarrow \delta'$ and $\alpha' \rightsquigarrow \delta'$. By induction on n , there exists $\delta \in F$ such that $\delta' \rightsquigarrow \delta$ and $\gamma \rightsquigarrow \delta$. Since then $\beta \rightsquigarrow \delta$, Claim 3 is proved. \square

Lemma 5.8. Assume $(*)$, and let $\alpha, \beta \in F$. Then $\alpha \sim \beta$ if and only if there is some $\gamma \in F$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$.

Proof. This is clear if α or β is zero, so assume $\alpha, \beta \neq 0$. If there exists $\gamma \in F$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$, it is clear from Lemma 5.5 that $\alpha \sim \beta$.

Conversely, assume that $\alpha \sim \beta$. Then there exists a finite string $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$, such that, for each $i = 0, \dots, n-1$, either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. We proceed by induction on n . If $n = 0$, then $\alpha = \beta$ and there is nothing to prove. Assume that $n > 0$ and the result is true for strings of length $n-1$, and let $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ be a string of length n . By the induction hypothesis, there is some $\lambda \in F$ such that $\alpha \rightarrow \lambda$ and $\alpha_{n-1} \rightarrow \lambda$. Now there are two cases to consider. If $\beta \rightarrow_1 \alpha_{n-1}$, then $\beta \rightarrow \lambda$ and we are done. Assume that $\alpha_{n-1} \rightarrow_1 \beta$. Then by Lemma 5.7, there is some $\gamma \in F$ such that $\beta \rightarrow \gamma$ and $\lambda \rightarrow \gamma$. Since $\alpha \rightarrow \lambda \rightarrow \gamma$, we get $\alpha \rightarrow \gamma$, and so the result is proven. \square

We are now ready to show the refinement property of $M(E, C, S)$ under the current finiteness hypothesis.

Proposition 5.9. Let (E, C) be a separated graph such that all the sets in C are finite, and let $S \subseteq C$. If Assumption $(*)$ holds, then the monoid $M(E, C, S)$ is a refinement monoid.

Proof. Because of Construction 5.3, it suffices to prove that $M(E, C)$ has refinement, i.e., we may assume that $S = C$.

Let $\alpha = \alpha_1 + \alpha_2 \sim \beta = \beta_1 + \beta_2$, with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in F$. By Lemma 5.8, there is some $\gamma \in F$ such that $\alpha \rightarrow \gamma$ and $\beta \rightarrow \gamma$. By Lemma 5.6, we can write $\gamma = \alpha'_1 + \alpha'_2 = \beta'_1 + \beta'_2$, with $\alpha_i \rightarrow \alpha'_i$ and $\beta_i \rightarrow \beta'_i$ for $i = 1, 2$. Since F is a free abelian monoid, F has the refinement property and so there are decompositions $\alpha'_i = \gamma_{i1} + \gamma_{i2}$ for $i = 1, 2$ such that $\beta'_j = \gamma_{1j} + \gamma_{2j}$ for $j = 1, 2$. Hence, $\alpha_i \sim \gamma_{i1} + \gamma_{i2}$ for $i = 1, 2$ and $\beta_j \sim \gamma_{1j} + \gamma_{2j}$ for $j = 1, 2$. The result follows. \square

We now return to the general case, where (E, C, S) is an arbitrary object of **SSGr**. Some of the above notation must then be modified.

Definition 5.10. For $v \in E^0$, let $C_{v, \text{fin}} := C_v \cap C_{\text{fin}}$ and $C_{v, \infty} := C_v \setminus C_{\text{fin}}$, and set

$$\mathcal{Z}_v = \mathcal{Z}_{E, C, v} := C_{v, \text{fin}} \sqcup \{\text{nonempty finite subsets of members of } C_{v, \infty}\}.$$

Then set $\mathcal{Z} = \mathcal{Z}_{E, C} := \bigsqcup_{v \in E^0} \mathcal{Z}_v$.

Next, let $Q^0 := \{q'_Z \mid Z \in \mathcal{Z} \setminus S\}$, and let F be the free abelian monoid on $E^0 \sqcup Q^0$. In F , set

$$\rho(Z) := \begin{cases} \mathbf{r}(Z) & (Z \in S) \\ \mathbf{r}(Z) + q'_Z & (Z \in \mathcal{Z} \setminus S) \end{cases}$$

for $Z \in \mathcal{Z}$, and set

$$\sigma(Z_1, Z_2) := \mathbf{r}(Z_2 \setminus Z_1) + q'_{Z_2} \quad (\emptyset \neq Z_1 \subsetneq Z_2 \in \mathcal{Z} \setminus C_{\text{fin}})$$

for nonempty sets $Z_1 \subsetneq Z_2$ from \mathcal{Z} . (Note that if $Z_1, Z_2 \in \mathcal{Z}$ with $Z_1 \subsetneq Z_2$, then necessarily $Z_1, Z_2 \notin C_{\text{fin}}$.) We identify $M(E, C, S)$ with F/\sim , where \sim is the congruence on F generated by the relations

$$v \sim \rho(Z) \quad \text{and} \quad q'_{Z_1} \sim \sigma(Z_1, Z_2)$$

for $v \in E^0$, $Z \in \mathcal{Z}_v$, and $\emptyset \neq Z_1 \subsetneq Z_2 \in \mathcal{Z} \setminus C_{\text{fin}}$.

Definition 5.11. [General case] For $\alpha, \beta \in F$, write $\alpha \rightsquigarrow_1 \beta$ to denote the following situation:

$\alpha = \sum_{i=1}^k u_i + \sum_{i=k+1}^m u_i$ and $\beta = \sum_{i=1}^k r_i + \sum_{i=k+1}^m u_i$ for some $u_i \in E^0 \sqcup Q^0$ such that for each $i = 1, \dots, k$, one of the following holds:

- (a) $u_i \in E^0 \setminus \text{Sink}(E)$ and $r_i = \rho(Z_i)$ for some $Z_i \in \mathcal{Z}_{u_i}$;
- (b) $u_i = q'_{Z_i} \in Q^0$ for some $Z_i \in \mathcal{Z} \setminus C_{\text{fin}}$, and $r_i = \sigma(Z_i, Z'_i)$ for some $Z'_i \in \mathcal{Z}$ with $Z_i \subsetneq Z'_i$.

Definition 5.12. [General case] **Assumption (*)**: For all $v \in E^0$ and $X, Y \in \mathcal{Z}_v$ such that $X \cup Y \notin \mathcal{Z}_v$, there exists $\gamma \in F$ such that $\rho(X) \rightsquigarrow_1 \gamma$ and $\rho(Y) \rightsquigarrow_1 \gamma$.

Note that there is no ambiguity in the two definitions of \rightsquigarrow_1 and (*).

Two conditions similar to Assumption (*) hold automatically, as follows.

Lemma 5.13. (a) If $v \in E^0$ and $X, Y \in \mathcal{Z}_v$ with $X \cup Y \in \mathcal{Z}_v$, there exists $\gamma \in F$ such that $\rho(X) \rightsquigarrow_1 \gamma$ and $\rho(Y) \rightsquigarrow_1 \gamma$.

(b) If $Z_1, Z_2, Z_3 \in \mathcal{Z}$ with $Z_1 \subsetneq Z_2$ and $Z_1 \subsetneq Z_3$, there exists $\gamma \in F$ such that $\sigma(Z_1, Z_2) \rightsquigarrow_1 \gamma$ and $\sigma(Z_1, Z_3) \rightsquigarrow_1 \gamma$.

Proof. (a) We may assume that $X \neq Y$. Then $Z := X \cup Y$ lies in $\mathcal{Z}_v \setminus C_{\text{fin}}$, and we observe that $\rho(X) \rightsquigarrow_1 r(X) + \sigma(X, Z) = \rho(Z)$ and similarly $\rho(Y) \rightsquigarrow_1 \rho(Z)$.

(b) Under the given hypotheses, $Z_2, Z_3 \notin C_{\text{fin}}$. Since they both contain Z_1 , they are not disjoint, and so they must both be subsets of some infinite member of C . Hence, $Z := Z_2 \cup Z_3$ lies in $\mathcal{Z} \setminus C_{\text{fin}}$, and we conclude that $\sigma(Z_1, Z_j) \rightsquigarrow_1 \sigma(Z_1, Z)$ for $j = 1, 2$. \square

Construction 5.14. As in Construction 5.3, we build a new graph \tilde{E} with appropriate finiteness conditions. First, let \tilde{E}^0 consist of E^0 together with new vertices w_Z for $Z \in \mathcal{Z} \setminus S$. Then let \tilde{E}^1 be the collection of the following four types of edges:

$$\begin{aligned} \varepsilon_{Z,e} &: v \rightarrow r(e), \text{ for } v \in E^0 \text{ and } e \in Z \in \mathcal{Z}_v; \\ f_Z &: v \rightarrow w_Z, \text{ for } v \in E^0 \text{ and } Z \in \mathcal{Z}_v \setminus S; \\ g_{Z,Z',e} &: w_Z \rightarrow r(e), \text{ for } \emptyset \neq Z \subsetneq Z' \in \mathcal{Z} \setminus C_{\text{fin}} \text{ and } e \in Z' \setminus Z; \\ h_{Z,Z'} &: w_Z \rightarrow w_{Z'}, \text{ for } \emptyset \neq Z \subsetneq Z' \in \mathcal{Z} \setminus C_{\text{fin}}. \end{aligned}$$

For each edge $e : v \rightarrow w$ in E , there is a unique smallest set $Z_e \in \mathcal{Z}_v$ containing e : namely, the unique set in C_v containing e , if this set is finite, or $\{e\}$ otherwise. If we identify e with the edge $\varepsilon_{Z_e,e}$ in \tilde{E} for each $e \in E^1$, then E becomes a subgraph of \tilde{E} .

Define partitions D_w of $s_{\tilde{E}}^{-1}(w)$ for $w \in \tilde{E}^0$ as follows. Set

$$X_Z := \begin{cases} \{\varepsilon_{Z,e} \mid e \in Z\} & (Z \in S) \\ \{\varepsilon_{Z,e} \mid e \in Z\} \sqcup \{f_Z\} & (Z \in \mathcal{Z} \setminus S) \end{cases}$$

for $Z \in \mathcal{Z}$, and define $D_v := \{X_Z \mid Z \in \mathcal{Z}_v\}$ for $v \in E^0$. Then set

$$Y_{Z,Z'} := \{g_{Z,Z',e} \mid e \in Z' \setminus Z\} \sqcup \{h_{Z,Z'}\} \quad (\emptyset \neq Z \subsetneq Z' \in \mathcal{Z} \setminus C_{\text{fin}}),$$

and define $D_{w_Z} := \{Y_{Z,Z'} \mid Z \subsetneq Z' \in \mathcal{Z}\}$ for $Z \in \mathcal{Z} \setminus C_{\text{fin}}$. For $Z \in C_{\text{fin}} \setminus S$, the vertex w_Z is a sink in \tilde{E} , and we set $D_{w_Z} = \emptyset$. We have now built a separated graph (\tilde{E}, D) , and all the sets in D are finite. Hence, (\tilde{E}, D, D) is an object in **SSGr**.

Comparing presentations, we see that there is a monoid isomorphism

$$M(E, C, S) \longrightarrow M(\tilde{E}, D, D)$$

sending $v \mapsto v$ for $v \in E^0$ and $q'_Z \mapsto w_Z$ for $Z \in \mathcal{Z} \setminus S$. There is an analogous isomorphism from F onto the free abelian monoid \tilde{F} on \tilde{E}^0 , and with its help, together with Lemma 5.13, we see that if Assumption $(*)$ holds for (E, C, S) , then it holds for (\tilde{E}, D, D) . Therefore we obtain the following theorem from Proposition 5.9.

Theorem 5.15. *Let (E, C) be a separated graph and $S \subseteq C_{\text{fin}}$. If Assumption $(*)$ holds, then the monoid $M(E, C, S)$ is a refinement monoid.*

Finally, let us apply Theorem 5.15 to the case of a non-separated graph E . The corresponding monoid $M(E)$ has only been defined in the literature in the row-finite case [9]; for the general case, we follow the pattern of Definition 4.1. Specifically, set $C = \bigsqcup_{v \in E^0} C_v$ and $S = C_{\text{fin}}$ where $C_v = \{s^{-1}(v)\}$ for $v \in E^0 \setminus \text{Sink}(E)$ and $C_v = \emptyset$ for $v \in \text{Sink}(E)$, and define $M(E) := M(E, C, S)$. With these choices of C and S , the collections \mathcal{Z}_v are closed under finite unions, and so Assumption $(*)$ holds vacuously. Therefore Theorem 5.15 has the following immediate consequence.

Corollary 5.16. *For any graph E , the monoid $M(E)$ is a refinement monoid.*

6. IDEAL LATTICES

The ideal lattices of Cohn-Leavitt algebras $CL_K(E, C, S)$ are typically much more complicated than those of Leavitt path algebras (of non-separated graphs), as is already clear from Proposition 2.10. In particular, the lattice of graded ideals of $CL_K(E, C, S)$ may contain many ideals not generated by vertices of E . We can, however, identify a sublattice of ideals of that type, namely the lattice of those ideals generated by idempotents. We prove that, in analogy with [39, Theorem 5.7], this lattice is isomorphic to a certain lattice $\mathcal{A}_{C,S}$ of *admissible pairs* (H, G) where $H \subseteq E^0$ and $G \subseteq C$ (see Definition 6.5 for the precise definition). These lattices, in turn, are isomorphic to the lattice of all order-ideals in the monoid $M(E, C, S)$. We can thus say that this lattice consists of “all the ideals of $CL_K(E, C, S)$ which can be detected by K-theory.” Further, we can derive graph-theoretic criteria for $M(E, C, S)$ to be simple (see Section 7).

See Definition 10.9 for the lattice of trace ideals of a ring A , denoted $\text{Tr}(A)$, and for the lattice of order-ideals of a monoid M , denoted $\mathcal{L}(M)$. We shall use the notation $\langle B \rangle$ to denote the order-ideal generated by a subset B of M .

Throughout this section, let (E, C, S) be a fixed object of **SSGr**, and recall the class $\mathcal{Z} = \mathcal{Z}_{E,C}$ from Definition 5.10. We will need the following idempotents $q_Z \in CL_K(E, C, S)$.

Definition 6.1. For any $v \in E^0$ and any nonempty finite subset Z of any $X \in C_v$, set

$$q_Z := v - \sum_{e \in Z} ee^*.$$

Of course, $q_Z = 0$ when $Z \in S$. The relations (SCK1) for X imply that the elements ee^* for $e \in X$ are pairwise orthogonal idempotents, and $ee^* \leq v$ for such e . Hence, q_Z is an idempotent, $q_Z \leq v$, and $q_Z \perp ee^*$ for all $e \in Z$. In fact, $q_Z e = e^* q_Z = 0$ for all $e \in Z$.

If $Z \notin S$, then we can ensure that v and the paths ee^* for $e \in Z$ are all reduced with respect to S in the sense of Definition 2.6, by simply choosing $e_X \in X \setminus Z$ in case $X \in S$. By Theorem 2.7, v and the ee^* for $e \in Z$ are K -linearly independent. Thus, $q_Z \neq 0$ when $Z \notin S$.

Proposition 6.2. *Let $A := CL_K(E, C, S)$. Then the trace ideals of A are precisely the idempotent-generated ideals, and the lattice isomorphism $\Phi : \mathcal{L}(\mathcal{V}(A)) \rightarrow \text{Tr}(A)$ of Proposition 10.10 can be expressed as*

$$\Phi(I) = \langle \text{idempotents } e \in A \mid [e] \in I \rangle.$$

Proof. Consider $J \in \text{Tr}(A)$, and let J' be the ideal of A generated by the idempotents in J . In order to prove that $J = J'$, it suffices to show that $\Psi(J) = \Psi(J')$, by Proposition 10.10. By Theorem 4.3, there is a monoid isomorphism $\Gamma : M(E, C, S) \rightarrow \mathcal{V}(A)$ sending $v \mapsto [v]$ for $v \in E^0$ and $q'_Z \mapsto [q_Z]$ for $Z \in \mathcal{Z} \setminus S$. Taking account of Definition 5.10, it follows that $\mathcal{V}(A)$ is generated by

$$\{[v] \mid v \in E^0\} \cup \{[q_Z] \mid Z \in \mathcal{Z} \setminus S\}.$$

Consequently, any $x \in \Psi(J)$ can be written $x = \sum_{i=1}^l [v_i] + \sum_{j=1}^m [q_{Z_j}]$ for some $v_i \in E^0$ and $Z_j \in \mathcal{Z} \setminus S$. Since $\Psi(J)$ is an order-ideal of $\mathcal{V}(A)$, we have $[v_i], [q_{Z_j}] \in \Psi(J)$ for all i, j , and hence $v_i, q_{Z_j} \in \Phi\Psi(J) = J$ for all i, j . Then $v_i, q_{Z_j} \in J'$ for all i, j , and consequently $x \in \Psi(J')$. Therefore $\Psi(J) = \Psi(J')$, as required.

Now let $I \in \mathcal{L}(\mathcal{V}(A))$. By what we have proved so far, the trace ideal $\Phi(I)$ is generated by the idempotents it contains. If $e \in A$ is idempotent, then $e \in \Phi(I)$ if and only if $[e] \in \Psi\Phi(I) = I$. Therefore $\Phi(I)$ is generated by the idempotents $e \in A$ such that $[e] \in I$. \square

Recall that for a non-separated row-finite graph E , the order-ideals of the monoid $M(E)$ correspond to hereditary saturated subsets of E^0 [9, Theorem 5.3]. In the case of $M(E, C, S)$, we require a version of saturation relative to the choices of C and S , as defined below.

Definition 6.3. Recall the relation \geq defined on E^0 by setting $v \geq w$ if and only if there is a path μ in E with $s(\mu) = v$ and $r(\mu) = w$. A subset H of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ always imply $w \in H$. The set H is called *saturated* if $r(s^{-1}(v)) \subseteq H$ implies $v \in H$ for any $v \in E^0$ which is not a sink or an infinite emitter.

A subset $H \subseteq E^0$ is (C, S) -saturated provided the following condition holds:

If $X \in S \cap C_v$ for some $v \in E^0$ and $r(X) \subseteq H$, then $v \in H$.

In case $S = C_{\text{fin}}$, we drop the reference to S and use the terminology C -saturated in place of (C, S) -saturated.

In working with the monoid $M(E, C, S)$, we continue to use the presentation given in Definition 5.10. Thus, $M(E, C, S)$ is generated by $E^0 \sqcup Q^0$ where $Q^0 = \{q'_Z \mid Z \in \mathcal{Z} \setminus S\}$ and $\mathcal{Z} = \mathcal{Z}_{E, C}$.

Lemma 6.4. *If I is an order-ideal of $M(E, C, S)$, then the set $H := \{v \in E^0 \mid v \in I\}$ is hereditary and (C, S) -saturated.*

Proof. Set $M := M(E, C, S)$.

Consider an edge $e : v \rightarrow w$ from E^1 such that $v \in H$. Then $e \in Z$ for some $Z \in \mathcal{Z}_v$, and $w \leq \mathbf{r}(Z) \leq \boldsymbol{\rho}(Z) = v$ in M . Since $v \in I$ and I is hereditary, we obtain $w \in I$ and so $w \in H$. This verifies that H is hereditary.

Next, consider $X \in S \cap C_v$ for some $v \in E^0$ such that $r(X) \subseteq H$. Then $v = \boldsymbol{\rho}(X) = \mathbf{r}(X) \in I$ and $v \in H$, showing that H is (C, S) -saturated. \square

We need to pair hereditary, (C, S) -saturated subsets of E^0 with certain subsets of C . The resulting pairs of sets are analogous to the *admissible pairs* with which Tomforde parametrized the graded ideals of $L_K(E)$ in [39, Theorem 5.7].

Definition 6.5. Let H be a hereditary, (C, S) -saturated subset of E^0 . For any subset $X \subseteq E^1$, define

$$X/H := X \cap r^{-1}(E^0 \setminus H).$$

(This notation follows that for the quotient graph E/H , which has vertex set $E^0 \setminus H$ and edge set $r_E^{-1}(E^0 \setminus H)$.) Set

$$\mathcal{G}(H) := \{X \in C \setminus S \mid X/H \text{ is nonempty and finite}\}.$$

Any $X \in \mathcal{G}(H)$ belongs to C_v for some $v \in E^0$, and there is some $e \in X$ with $r(e) \notin H$, so $v = s(e) \notin H$ because H is hereditary. Thus, $\mathcal{G}(H) \subseteq \bigsqcup_{v \in E^0 \setminus H} C_v$. We record this in the form $\mathcal{G}(H) \cap C[H] = \emptyset$, where

$$C[H] := \bigsqcup_{v \in H} C_v.$$

Note that $X/H \in \mathcal{Z} \setminus S$ for all $X \in \mathcal{G}(H) \cap C_\infty$, and so $q'_{X/H} \in Q^0$ for all $X \in \mathcal{G}(H) \cap C_\infty$.

Next, let $\mathcal{A}_{C,S}$ denote the set of all pairs (H, G) where H is a hereditary, (C, S) -saturated subset of E^0 and G is a subset of $\mathcal{G}(H)$. Define a relation \leq on $\mathcal{A}_{C,S}$ as follows:

$$(H_1, G_1) \leq (H_2, G_2) \iff H_1 \subseteq H_2 \text{ and } G_1 \subseteq G_2 \sqcup C[H_2].$$

Observe that $(\mathcal{A}_{C,S}, \leq)$ is a partially ordered set, with minimum element (\emptyset, \emptyset) and maximum element (E^0, \emptyset) .

We claim that any nonempty family $((H_i, G_i))_{i \in I}$ of elements of $\mathcal{A}_{C,S}$ has an infimum in $\mathcal{A}_{C,S}$, namely the pair

$$(H, G) := \left(\bigcap_{i \in I} H_i, \mathcal{G}\left(\bigcap_{i \in I} H_i\right) \cap \bigcap_{i \in I} (G_i \sqcup C[H_i]) \right).$$

It is clear that $(H, G) \in \mathcal{A}_{C,S}$ and $(H, G) \leq (H_i, G_i)$ for all i . If $(H', G') \in \mathcal{A}_{C,S}$ and $(H', G') \leq (H_i, G_i)$ for all i , then clearly $H' \subseteq H$. Consider any $X \in G' \setminus C[H]$. Then $X \in C_v$ for some $v \in E^0 \setminus H$, so $v \notin H_j$ for some $j \in I$, and $X \notin C[H_j]$. Since $(H', G') \leq (H_j, G_j)$, it follows that $X \in G_j$. Hence, X/H_j is nonempty, whence X/H is nonempty. On the other hand, $X \in G'$ implies that X/H' is finite, whence X/H is finite. Thus, $X \in \mathcal{G}(H)$. We also have $X \in G_i \sqcup C[H_i]$ for all i , because $(H', G') \leq (H_i, G_i)$ for all i , and consequently $X \in G$. This shows that $G' \subseteq G \sqcup C[H]$, proving that $(H', G') \leq (H, G)$, and verifying that (H, G) is indeed the infimum of the family $((H_i, G_i))_{i \in I}$ in $\mathcal{A}_{C,S}$.

Therefore, $\mathcal{A}_{C,S}$ is a complete lattice.

Definition 6.6. If I is any order-ideal of $M(E, C, S)$, set $\psi(I) := (H, G)$ where

$$H := \{v \in E^0 \mid v \in I\}$$

$$G := \{X \in \mathcal{G}(H) \cap C_{\text{fin}} \mid q'_X \in I\} \sqcup \{X \in \mathcal{G}(H) \cap C_\infty \mid q'_{X/H} \in I\}.$$

As we have seen in Lemma 6.4, H is a hereditary, (C, S) -saturated subset of E^0 , and so $(H, G) \in \mathcal{A}_{C,S}$.

Conversely, for any $(H, G) \in \mathcal{A}_{C,S}$, let $I(H, G)$ denote the submonoid of $M(E, C, S)$ generated by the set

$$H \sqcup \{q'_X \mid X \in C_{\text{fin}} \cap G\} \sqcup \{q'_{X/H} \mid X \in C_\infty \cap G\}.$$

Lemma 6.7. *If I is any order-ideal of $M(E, C, S)$ and $\psi(I) = (H, G)$, then $I = \langle I(H, G) \rangle$.*

Proof. It is clear that $I(H, G) \subseteq I$, whence $\langle I(H, G) \rangle \subseteq I$. Now consider a nonzero element $x \in I$. Then $x = \sum_i v_i + \sum_j q'_{Z_j}$ for some $v_i \in E^0$ and some $Z_j \in \mathcal{Z} \setminus S$. Each $v_i, q'_{Z_j} \in I$ because I is an order-ideal, and so to prove that $x \in \langle I(H, G) \rangle$, it is enough to show that $v, q'_Z \in \langle I(H, G) \rangle$ for all $v \in E^0$ and $Z \in \mathcal{Z} \setminus S$ such that $v, q'_Z \in I$.

If $v \in E^0 \cap I$, then $v \in H$ by definition of H , whence $v \in I(H, G)$.

Now let $Z \in \mathcal{Z} \setminus S$ such that $q'_Z \in I$. Then $Z \subseteq X$ for some $X \in C_v \setminus S$ and $v \in E^0$. If $r(Z) \subseteq H$, then $\mathbf{r}(Z) \in I$ and so

$$v = \boldsymbol{\rho}(Z) = \mathbf{r}(Z) + q'_Z \in I,$$

whence $v \in H \subseteq I(H, G)$. Since $q'_Z \leq v$, it follows that $q'_Z \in \langle I(H, G) \rangle$ in this case.

Assume now that $r(Z) \not\subseteq H$, so that Z/H and X/H are nonempty. If X is finite, then $X = Z \in \mathcal{G}(H)$ and $q'_X = q'_Z \in I$, so $X \in G$ by definition of G and $q'_Z = q'_X \in I(H, G)$. If X is infinite and X/H is finite, then $X \in \mathcal{G}(H)$ and

$$\mathbf{r}((X/H) \setminus Z) = \mathbf{r}([(X/H) \cup Z] \setminus Z) \leq \boldsymbol{\sigma}(Z, (X/H) \cup Z) = q'_Z \in I.$$

In this case, it follows that $\mathbf{r}((X/H) \setminus Z) \in I$ and so $r((X/H) \setminus Z) \subseteq H$, whence $X/H \subseteq Z$. Since $r(Z \setminus (X/H)) \subseteq H$, we get

$$q'_{X/H} = \boldsymbol{\sigma}(X/H, Z) = \mathbf{r}(Z \setminus (X/H)) + q'_Z \in I,$$

so that $X \in G$ by definition of G , and then $q'_{X/H} \in I(H, G)$. Since $q'_Z \leq q'_{X/H}$, we get $q'_Z \in \langle I(H, G) \rangle$.

Finally, assume that X and X/H are both infinite. Then there exists $f \in (X/H) \setminus Z$, and we have

$$r(f) \leq \boldsymbol{\sigma}(Z, Z \sqcup \{f\}) = q'_Z \in I.$$

But this implies $r(f) \in I$ and so $r(f) \in H$, which is a contradiction to the assumption that $f \in X/H$. Thus, $q'_Z \in \langle I(H, G) \rangle$ in all cases, as required. \square

We define the *kernel* of a monoid homomorphism $\tau : M_1 \rightarrow M_2$ just as for abelian groups: $\ker \tau := \tau^{-1}(\{0\})$. This is always a submonoid of M_1 , but if M_2 is conical, then $\ker \tau$ is an order-ideal of M_1 .

Construction 6.8. Let $(H, G) \in \mathcal{A}_{C,S}$ and define the order-ideal $I := \langle I(H, G) \rangle$ in $M := M(E, C, S)$. We construct an object $(\tilde{E}, \tilde{C}, \tilde{S})$ in \mathbf{SSGr} and a homomorphism π from M to $\tilde{M} := M(\tilde{E}, \tilde{C}, \tilde{S})$ such that $I = \ker \pi$.

Let \tilde{E} be the quotient graph E/H , that is, the subgraph of E with

$$\tilde{E}^0 = E^0 \setminus H \quad \text{and} \quad \tilde{E}^1 = r_E^{-1}(E^0 \setminus H) = E^1 \setminus H.$$

(Since H is hereditary, $s(e) \notin H$ for all $e \in \tilde{E}^1$.) For $v \in \tilde{E}^0$, set

$$\tilde{C}_v := \{X/H \mid X \in C_v \text{ and } X/H \neq \emptyset\},$$

which is a partition of $s_{\tilde{E}}^{-1}(v)$. Now set $\tilde{C} := \bigsqcup_{v \in \tilde{E}^0} \tilde{C}_v$. This gives us a separated graph (\tilde{E}, \tilde{C}) . Further, set

$$\tilde{S} := \{X/H \mid X \in S \text{ and } X/H \neq \emptyset\} \sqcup \{X/H \mid X \in G\}.$$

Clearly, $X/H \in \tilde{C}_{\text{fin}}$ when $X \in S$ and $X/H \neq \emptyset$. By definition of $\mathcal{G}(H)$, the set X/H is nonempty and finite for any $X \in G$, whence $X/H \in \tilde{C}_{\text{fin}}$. Thus, $\tilde{S} \subseteq \tilde{C}_{\text{fin}}$, and therefore $(\tilde{E}, \tilde{C}, \tilde{S})$ is an object in **SSGr**. Observe that, for $X \in C \setminus S$, we have

$$X \in \mathcal{G}(H) \iff X/H \in \tilde{C}_{\text{fin}}.$$

We next define elements $\tilde{v}, \tilde{q}_Z \in \tilde{M}$ for $v \in E^0$ and $Z \in \mathcal{Z} \setminus S$. For $v \in E^0$, set

$$\tilde{v} := \begin{cases} v & \text{if } v \notin H \\ 0 & \text{if } v \in H. \end{cases}$$

Next, assume that $Z \in \mathcal{Z} \setminus S$; then $Z \subseteq X$ for some $X \in C_v \setminus S$ and $v \in E^0$. We distinguish several cases:

- (1) If $v \in H$, set $\tilde{q}_Z := 0$.
- (2) If $v \notin H$ and $X \in G$, set $\tilde{q}_Z := \mathbf{r}((X \setminus Z)/H)$. (In case X is finite, $X = Z$ and $\tilde{q}_Z = 0$.)

$$(3) \text{ If } v \notin H \text{ and } X \notin G, \text{ set } \tilde{q}_Z := \begin{cases} v & \text{if } r(Z) \subseteq H \\ q'_{Z/H} & \text{if } r(Z) \not\subseteq H \text{ and } X \notin \mathcal{G}(H) \\ q'_{X/H} + \mathbf{r}((X \setminus Z)/H) & \text{if } r(Z) \not\subseteq H \text{ and } X \in \mathcal{G}(H). \end{cases}$$

Note that if $v \notin H$ and $X \in \mathcal{G}(H) \setminus G$, then $\tilde{q}_Z = q'_{X/H} + \mathbf{r}((X \setminus Z)/H)$ whether or not $r(Z) \subseteq H$.

Claim 1: There is a homomorphism $\pi : M \rightarrow \tilde{M}$ sending $v \mapsto \tilde{v}$ for all $v \in E^0$ and $q'_Z \mapsto \tilde{q}_Z$ for all $Z \in \mathcal{Z} \setminus S$. To see this, we need to verify that the elements \tilde{v} and \tilde{q}_Z satisfy the defining relations of the elements $v, q'_Z \in E^0 \sqcup Q^0$. In order to write these relations compactly, it will be convenient to use the notation

$$\tilde{\mathbf{r}}(W) := \sum_{e \in W} \widetilde{r(e)} \in \tilde{M}$$

for subsets $W \subseteq E^1$. Note that if we view W/H as a subset of \tilde{E}^1 , then $\tilde{\mathbf{r}}(W) = \mathbf{r}(W/H)$, since $\widetilde{r(e)} = 0$ for $e \in W \setminus (W/H)$.

Suppose that $v \in E^0$ and $Z \in \mathcal{Z}_v$. Then $Z \subseteq X$ for some $X \in C_v$. If $v \in H$, then $r(Z) \subseteq H$ because H is hereditary, and we get

$$\tilde{v} = 0 = \begin{cases} \tilde{\mathbf{r}}(Z) & \text{if } Z \in S \\ \tilde{\mathbf{r}}(Z) + \tilde{q}_Z & \text{if } Z \notin S. \end{cases}$$

If $v \notin H$ and $Z \in S$, we have $Z = X$ and $r(X) \not\subseteq H$ because H is (C, S) -saturated, whence

$$\tilde{v} = v = \boldsymbol{\rho}(Z/H) = \mathbf{r}(Z/H) = \tilde{\mathbf{r}}(Z).$$

So, we may assume that $v \notin H$ and $Z \notin S$. Then also $X \notin S$.

In case (2), $X/H \in \tilde{S}$ and

$$\tilde{v} = v = \boldsymbol{\rho}(X/H) = \mathbf{r}(X/H) = \tilde{q}_Z + \mathbf{r}(Z/H) = \tilde{\mathbf{r}}(Z) + \tilde{q}_Z.$$

In case (3) with $r(Z) \subseteq H$, we have

$$\tilde{v} = v = \tilde{q}_Z = \tilde{\mathbf{r}}(Z) + \tilde{q}_Z.$$

In case (3) with $r(Z) \not\subseteq H$ and $X \notin \mathcal{G}(H)$, we have

$$\tilde{v} = v = \boldsymbol{\rho}(Z/H) = \mathbf{r}(Z/H) + q'_{Z/H} = \tilde{\mathbf{r}}(Z) + \tilde{q}_Z.$$

Finally, in case (3) with $r(Z) \not\subseteq H$ and $X \in \mathcal{G}(H)$, we have

$$\tilde{v} = v = \boldsymbol{\rho}(X/H) = \mathbf{r}(X/H) + q'_{X/H} = \mathbf{r}(Z/H) + \mathbf{r}((X \setminus Z)/H) + q'_{X/H} = \tilde{\mathbf{r}}(Z) + \tilde{q}_Z.$$

Thus, the \tilde{v} and \tilde{q}_Z satisfy all the required relations of the type $v = \boldsymbol{\rho}(Z)$.

Assume now that $\emptyset \neq Z_1 \subsetneq Z_2 \in \mathcal{Z}_v \setminus C_{\text{fin}}$ for some $v \in E^0$. Then $Z_2 \subseteq X$ for some $X \in C_{v,\infty}$. If $v \in H$, then $r(Z_2) \subseteq H$ and

$$\tilde{q}_{Z_1} = 0 = \tilde{\mathbf{r}}(Z_2 \setminus Z_1) + \tilde{q}_{Z_2}.$$

So, we may assume that $v \notin H$.

In case (2),

$$\tilde{q}_{Z_1} = \mathbf{r}((X \setminus Z_1)/H) = \mathbf{r}((Z_2 \setminus Z_1)/H) + \mathbf{r}((X \setminus Z_2)/H) = \tilde{\mathbf{r}}(Z_2 \setminus Z_1) + \tilde{q}_{Z_2}.$$

If $X \in \mathcal{G}(H) \setminus G$, then

$$\begin{aligned} \tilde{q}_{Z_1} &= q'_{X/H} + \mathbf{r}((X \setminus Z_1)/H) = q'_{X/H} + \mathbf{r}((X \setminus Z_2)/H) + \mathbf{r}((Z_2 \setminus Z_1)/H) \\ &= \tilde{q}_{Z_2} + \tilde{\mathbf{r}}(Z_2 \setminus Z_1). \end{aligned}$$

Hence, in dealing with case (3) we may assume that $X \notin \mathcal{G}(H)$. In case (3) with $r(Z_2) \subseteq H$, we have

$$\tilde{q}_{Z_1} = v = \mathbf{r}((Z_2 \setminus Z_1)/H) + v = \tilde{\mathbf{r}}(Z_2 \setminus Z_1) + \tilde{q}_{Z_2}.$$

In case (3) with $r(Z_1) \subseteq H$ but $r(Z_2) \not\subseteq H$, we have

$$\tilde{q}_{Z_1} = v = \boldsymbol{\rho}(Z_2/H) = \mathbf{r}(Z_2/H) + q'_{Z_2/H} = \tilde{\mathbf{r}}(Z_2 \setminus Z_1) + \tilde{q}_{Z_2}.$$

Finally, in case (3) with $r(Z_1) \not\subseteq H$, we have

$$\tilde{q}_{Z_1} = q'_{Z_1/H} = \boldsymbol{\sigma}(Z_1/H, Z_2/H) = \mathbf{r}((Z_2 \setminus Z_1)/H) + q'_{Z_2/H} = \tilde{\mathbf{r}}(Z_2 \setminus Z_1) + \tilde{q}_{Z_2}.$$

This verifies that the \tilde{v} and \tilde{q}_Z satisfy all the required relations of the type $q'_{Z_1} = \boldsymbol{\sigma}(Z_1, Z_2)$.

Therefore Claim 1 is established. Since \tilde{M} is conical, $\ker \pi$ is an order-ideal of M .

Claim 2: $I \subseteq \ker \pi$. Since $\ker \pi$ is an order-ideal, it suffices to show that $I(H, G) \subseteq \ker \pi$.

For $v \in H$, we have $\pi(v) = \tilde{v} = 0$. For $X \in G \cap C_{\text{fin}}$, we have $X \in C_v$ for some $v \in E^0 \setminus H$, and $\pi(q'_X) = \tilde{q}_X = 0$ by case (2) above. If $X \in G \cap C_\infty$, then $X/H \in \mathcal{Z} \setminus S$ and we have

$\pi(q'_{X/H}) = \tilde{q}_{X/H} = 0$, using case (2) again. This verifies that all the generators of $I(H, G)$ lie in $\ker \pi$.

Claim 3: $\psi(\ker \pi) = (H, G)$.

Let $(H', G') = \psi(\ker \pi)$. It is clear from the construction of π that

$$H' = E^0 \cap (\ker \pi) = H.$$

In view of Claim 2, it immediately follows that $G \subseteq G'$.

Consider $X \in \mathcal{G}(H)$. Then $X \in C_v$ for some $v \in E^0 \setminus H$. If X is finite and $X \notin G$, then $\pi(q'_X) = \tilde{q}_X \neq 0$ by (3). Hence, $q'_X \notin \ker \pi$ and so $X \notin G'$. If X is infinite and $X \notin G$, then $\pi(q'_{X/H}) = \tilde{q}_{X/H} \neq 0$ by (3). Thus $q'_{X/H} \notin \ker \pi$ in this case, and again $X \notin G'$. This verifies that $G' = G$, establishing Claim 3.

Claim 4: $\ker \pi = I$, and therefore $\psi(I) = (H, G)$.

In view of Claim 3, Lemma 6.7 implies that $\ker \pi = \langle I(H, G) \rangle = I$, which establishes Claim 4.

Theorem 6.9. *Let (E, C) be a separated graph, and S a subset of C_{fin} . Then there are mutually inverse lattice isomorphisms*

$$\varphi: \mathcal{A}_{C,S} \longrightarrow \mathcal{L}(M(E, C, S)) \quad \text{and} \quad \psi: \mathcal{L}(M(E, C, S)) \longrightarrow \mathcal{A}_{C,S},$$

where $\varphi(H, G) = \langle I(H, G) \rangle$ for $(H, G) \in \mathcal{A}_{C,S}$ and ψ is defined as in Definition 6.6.

Proof. The maps φ and ψ are well defined by construction, and Lemma 6.7 shows that $\varphi\psi$ is the identity map on $\mathcal{L}(M(E, C, S))$. That $\psi\varphi$ is the identity map on $\mathcal{A}_{C,S}$ follows from Claim 4 of Construction 6.8. It thus remains to show that φ and ψ are order-preserving, since then they are isomorphisms of posets, and therefore lattice isomorphisms.

Suppose $I_1 \subseteq I_2$ are order-ideals of $M(E, C, S)$ and $(H_j, G_j) = \psi(I_j)$ for $j = 1, 2$. Obviously $H_1 \subseteq H_2$. Let $X \in G_1$; then $X \in C_v$ for some $v \in E^0 \setminus H_1$. First suppose that $X \in \mathcal{G}(H_1) \cap C_{\text{fin}}$ and $q'_X \in I_1$. If $X \in \mathcal{G}(H_2)$, then $X \in G_2$. Otherwise, $r(X) \subseteq H_2$ and so $\mathbf{r}(X) \in I_2$, whence $v = \boldsymbol{\rho}(X) \in I_2$, yielding $v \in H_2$ and $X \in C[H_2]$. Now suppose that $X \in \mathcal{G}(H_1) \cap C_\infty$ and $q'_{X/H_1} \in I_1$. If $X \in \mathcal{G}(H_2)$, then q'_{X/H_2} is defined and

$$q'_{X/H_2} = \boldsymbol{\sigma}(X/H_2, X/H_1) = \mathbf{r}(X \cap r^{-1}(H_2 \setminus H_1)) + q'_{X/H_1} \in I_2,$$

so $X \in G_2$. Otherwise, $r(X) \subseteq H_2$ and so $\mathbf{r}(X/H_1) \in I_2$, whence $v = \boldsymbol{\rho}(X/H_1) \in I_2$, again yielding $X \in C[H_2]$. We have now shown that $G_1 \subseteq G_2 \sqcup C[H_2]$, and so $(H_1, G_1) \leq (H_2, G_2)$. Therefore ψ is order-preserving.

Finally, let (H_1, G_1) and (H_2, G_2) be any elements of $\mathcal{A}_{C,S}$ such that $(H_1, G_1) \leq (H_2, G_2)$. Obviously $H_1 \subseteq I(H_2, G_2)$. Consider $X \in G_1 \cap C_{\text{fin}}$. If $X \in G_2$, then $q'_X \in I(H_2, G_2)$ by definition of $I(H_2, G_2)$. If $X \in C_v$ for some $v \in H_2$, then

$$q'_X \leq \boldsymbol{\rho}(X) = v \in I(H_2, G_2)$$

and so $q'_X \in \langle I(H_2, G_2) \rangle$. Now consider $X \in G_1 \cap C_\infty$. If $X \in G_2$, then $q'_{X/H_2} \in I(H_2, G_2)$. Since

$$q'_{X/H_1} \leq \boldsymbol{\sigma}(X/H_2, X/H_1) = q'_{X/H_2},$$

it follows that $q'_{X/H_1} \in \langle I(H_2, G_2) \rangle$. If $X \in C_v$ for some $v \in H_2$, then

$$q'_{X/H_1} \leq \rho(X/H_1) = v \in I(H_2, G_2)$$

and again $q'_{X/H_1} \in \langle I(H_2, G_2) \rangle$. Thus, all the generators of $I(H_1, G_1)$ lie in $\varphi(H_2, G_2)$, and we conclude that $\varphi(H_1, G_1) \subseteq \varphi(H_2, G_2)$. Therefore φ is order-preserving. \square

In case $C = C_{\text{fin}} = S$, the lattice $\mathcal{A}_{C,S}$ consists of pairs (H, \emptyset) , and so $\mathcal{A}_{C,S}$ is naturally isomorphic to the lattice of hereditary, C -saturated subsets of E^0 . We thus obtain the following corollary of Theorem 6.9.

Corollary 6.10. *Let (E, C) be a separated graph such that all the sets in C are finite, and let \mathcal{H} be the lattice of hereditary, C -saturated subsets of E^0 . Then there are mutually inverse lattice isomorphisms*

$$\varphi : \mathcal{H} \longrightarrow \mathcal{L}(M(E, C)) \quad \text{and} \quad \psi : \mathcal{L}(M(E, C)) \longrightarrow \mathcal{H},$$

where $\varphi(H) = \langle H \rangle$ for $H \in \mathcal{H}$ and $\psi(I) = \{v \in E^0 \mid v \in I\}$ for $I \in \mathcal{L}(M(E, C))$. \square

The combination of Theorem 6.9 with Propositions 10.10 and 6.2 yields the following description of the lattice of trace ideals in $CL_K(E, C, S)$.

Theorem 6.11. *Let (E, C) be a separated graph, S a subset of C_{fin} , and $A := CL_K(E, C, S)$. Then there are mutually inverse lattice isomorphisms*

$$\xi : \mathcal{A}_{C,S} \longrightarrow \text{Tr}(A) \quad \text{and} \quad \theta : \text{Tr}(A) \longrightarrow \mathcal{A}_{C,S},$$

given by the rules

$$\begin{aligned} \xi(H, G) &:= \langle H \sqcup \{q_X \mid X \in G \cap C_{\text{fin}}\} \sqcup \{q_{X/H} \mid X \in G \cap C_{\infty}\} \rangle \\ \theta(J) &:= (E^0 \cap J, \{X \in \mathcal{G}(E^0 \cap J) \cap C_{\text{fin}} \mid q_X \in J\} \\ &\quad \sqcup \{X \in \mathcal{G}(E^0 \cap J) \cap C_{\infty} \mid q_{X/H} \in J\}). \end{aligned}$$

Proof. Set $M := M(E, C, S)$, and let $\Gamma := \Gamma(E, C, S) : M \rightarrow \mathcal{V}(A)$ be the monoid isomorphism given by Theorem 4.3. We shall also use Γ to denote the induced lattice isomorphism $\mathcal{L}(M) \rightarrow \mathcal{L}(\mathcal{V}(A))$. Due to Theorem 6.9 and Proposition 10.10, we have mutually inverse lattice isomorphisms

$$\Phi\Gamma\varphi : \mathcal{A}_{C,S} \longrightarrow \text{Tr}(A) \quad \text{and} \quad \psi\Gamma^{-1}\Psi : \text{Tr}(A) \longrightarrow \mathcal{A}_{C,S}.$$

It is clear that $\psi\Gamma^{-1}\Psi(J) = \theta(J)$ for all $J \in \text{Tr}(A)$. For $(H, G) \in \mathcal{A}_{C,S}$, we note, using Proposition 6.2, that

$$\Phi\Gamma\varphi(H, G) = \langle \text{idempotents } e \in A \mid [e] \in J(H, G) \rangle,$$

where $J(H, G)$ is the order-ideal of $\mathcal{V}(A)$ generated by the set

$$\{[v] \mid v \in H\} \cup \{[q_X] \mid X \in G \cap C_{\text{fin}}\} \cup \{[q_{X/H}] \mid X \in G \cap C_{\infty}\}.$$

Note that $\xi(H, G) \subseteq \Phi\Gamma\varphi(H, G)$. If e is an idempotent in A such that $[e] \in J(H, G)$, then

$$[e] \leq \sum_{i=1}^l [v_i] + \sum_{j=1}^m [q_{X_j}] + \sum_{k=1}^n [q_{Y_k/H}]$$

for some $v_i \in H$, $X_j \in G \cap C_{\text{fin}}$, and $Y_k \in G \cap C_{\infty}$. Consequently, e is equivalent to some idempotent

$$e' \leq \text{diag}(v_1, \dots, v_l, q_{X_1}, \dots, q_{X_m}, q_{Y_1/H}, \dots, q_{Y_n/H}),$$

from which it follows that e lies in the ideal generated by these v_i , q_{X_j} , and $q_{Y_k/H}$. Therefore, $\Phi\Gamma\varphi(H) = \xi(H, G)$. \square

The lattices $\text{Tr}(CL_K(E, C, S))$ and $\mathcal{L}(M(E, C, S))$ are also isomorphic to the lattice of hereditary, (C, S) -saturated subsets of a graph $XE \supseteq E$, which is a “slimmer” version of the graph in Construction 5.14, as follows.

Definition 6.12. The (C, S) -extension of E is the graph $XE = X_{C,S}(E)$ with vertex set

$$XE^0 := E^0 \sqcup (\mathcal{Z} \setminus S)$$

and with edge set XE^1 consisting of the following four types of edges:

All edges in E^1 ;

One edge $v \rightarrow Z$ for each $v \in E^0$ and $Z \in \mathcal{Z}_v \setminus S$;

One edge $Z \rightarrow r(f)$ for each $Z \in \mathcal{Z} \setminus C_{\text{fin}}$ and $f \in E^1 \setminus Z$ such that $Z \sqcup \{f\} \in \mathcal{Z}$;

One edge $Z \rightarrow Z'$ for each $Z, Z' \in \mathcal{Z} \setminus C_{\text{fin}}$ such that $Z \subsetneq Z'$.

Of course, if all the sets in C are finite, then XE^1 contains only edges of the first two types.

Definition 6.13. A subset $H \subseteq XE^0$ is (C, S) -saturated provided the following three conditions hold:

- (1) If $X \in S \cap C_v$ for some $v \in E^0$ and $r(X) \subseteq H$, then $v \in H$.
- (2) If $Z \in \mathcal{Z}_v \setminus S$ for some $v \in E^0$ with $Z \in H$ and $r(Z) \subseteq H$, then $v \in H$.
- (3) If $\emptyset \neq Z \subsetneq Z' \in \mathcal{Z} \setminus C_{\text{fin}}$ with $Z' \in H$ and $r(Z' \setminus Z) \subseteq H$, then $Z \in H$.

Let $\mathcal{H}_{C,S}$ denote the family of all hereditary, (C, S) -saturated subsets of XE^0 . This family is closed under arbitrary intersections, and hence it is a complete lattice with respect to inclusion.

Remark 6.14. Let (E, C) be a separated graph, and S a subset of C_{fin} . There are lattice isomorphisms as indicated below. We omit the proofs, which are similar to those of Theorems 6.9 and 6.11.

First, there are mutually inverse lattice isomorphisms

$$\varphi: \mathcal{H}_{C,S} \longrightarrow \mathcal{L}(M(E, C, S)) \quad \text{and} \quad \psi: \mathcal{L}(M(E, C, S)) \longrightarrow \mathcal{H}_{C,S},$$

given by the rules

$$\begin{aligned} \varphi(H) &:= \langle (E^0 \cap H) \cup \{q'_Z \mid Z \in (\mathcal{Z} \setminus S) \cap H\} \rangle \\ \psi(I) &:= (E^0 \cap I) \sqcup \{Z \in \mathcal{Z} \setminus S \mid q'_Z \in I\}. \end{aligned}$$

Second, there are mutually inverse lattice isomorphisms

$$\xi : \mathcal{H}_{C,S} \longrightarrow \text{Tr}(CL_K(E, C, S)) \quad \text{and} \quad \theta : \text{Tr}(CL_K(E, C, S)) \longrightarrow \mathcal{H}_{C,S},$$

given by the rules

$$\begin{aligned} \xi(H) &:= \langle (E^0 \cap H) \cup \{q_Z \mid Z \in (\mathcal{Z} \setminus S) \cap H\} \rangle \\ \theta(J) &:= \{v \in E^0 \mid v \in J\} \sqcup \{Z \in \mathcal{Z} \setminus S \mid q_Z \in J\}. \end{aligned}$$

7. SIMPLICITY

The description of the lattices $\mathcal{L}(M(E, C, S))$ in the previous section allows us to develop criteria for simplicity of the monoids $M(E, C, S)$. In view of Proposition 10.10, we thus obtain criteria for $CL_K(E, C, S)$ to be “trace-simple” in the sense that the only trace ideals of this algebra are 0 and $CL_K(E, C, S)$.

In general, a monoid M is *simple* provided M has precisely two order-ideals, namely M itself and the group of units of M . In the conical case (as for $M(E, C, S)$), simplicity means that M is nonzero and its only order-ideals are $\{0\}$ and M .

Theorem 7.1. *Let (E, C) be a separated graph, and S a subset of C_{fin} . The following conditions are equivalent.*

- (1) *The only trace ideals of $CL_K(E, C, S)$ are 0 and $CL_K(E, C, S)$.*
- (2) *$M(E, C, S)$ is a simple monoid.*
- (3) (a) *$S = C_{\text{fin}}$, and*
 (b) *The only hereditary, (C, S) -saturated subsets of E^0 are \emptyset and E^0 .*

Proof. The equivalence of (1) and (2) is immediate from Proposition 10.10, given that $M(E, C, S)$ is nonzero and conical (Lemma 4.2). Next, observe that for the hereditary, (C, S) -saturated subsets $\emptyset, E^0 \subseteq E^0$, we have $\mathcal{G}(\emptyset) = C_{\text{fin}} \setminus S$ and $\mathcal{G}(E^0) = \emptyset$.

(2) \implies (3): Assume that $M(E, C, S)$ is a simple monoid. By Theorem 6.9, the only members of $\mathcal{A}_{C,S}$ are (\emptyset, \emptyset) and (E^0, \emptyset) . Since $(\emptyset, \{X\}) \in \mathcal{A}_{C,S}$ for any $X \in C_{\text{fin}} \setminus S$, condition (a) follows. Further, since $(H, \emptyset) \in \mathcal{A}_{C,S}$ for any hereditary, (C, S) -saturated subset H of E^0 , condition (b) follows as well.

(3) \implies (2): If (a) and (b) hold, the only members of $\mathcal{A}_{C,S}$ are (\emptyset, \emptyset) and (E^0, \emptyset) . Theorem 6.9 then implies that $M(E, C, S)$ is simple. \square

Corollary 7.2. *Let E be any (non-separated) graph. Then $M(E)$ is a simple monoid if and only if the only hereditary, saturated subsets of E^0 are \emptyset and E^0 .*

Proof. We have $M(E) = M(E, C, S)$ where $S = C_{\text{fin}}$ and C is the union of the singleton collections $\{s^{-1}(v)\}$ for non-sinks $v \in E^0$. With these choices of C and S , a subset of E^0 is (C, S) -saturated if and only if it is saturated. Therefore the corollary follows immediately from Theorem 7.1. \square

Remark 7.3. Note that if $C = C_{\text{fin}} = S$, then every (C, S) -saturated subset of E^0 is saturated. Consequently, if $M(E)$ is simple, then so is $M(E, C, S)$. Namely, simplicity of $M(E)$

implies that \emptyset and E^0 are the only hereditary, saturated subsets of E^0 (Corollary 7.2), and so these are the only hereditary, (C, S) -saturated subsets of E^0 .

Conversely, though, it is very easy to produce examples (E, C, S) with $C = C_{\text{fin}} = S$ such that $M(E, C, S)$ is simple but $M(E)$ is not simple. For example, consider the following graph E :

$$e \curvearrowright v \xrightarrow{f} w$$

Let $C := C_v := \{\{e\}, \{f\}\}$, and take $S = C$. Since $\{f\} \in S$, any (C, S) -saturated subset of E^0 which contains w must also contain v . Thus, $\mathcal{A}_{C,S} = \{(\emptyset, \emptyset), (E^0, \emptyset)\}$, and so $M(E, C, S)$ is simple. This can be verified directly: $M(E, C, S)$ is generated by v and w with the relations $v = v = w$, and so $M(E, C, S) \cong \mathbb{Z}^+$. On the other hand, $\{w\}$ is a hereditary, saturated subset of E^0 , whence $M(E)$ is not simple. In fact, $M(E)$ is generated by v and w with the sole relation $v = v + w$, and so w generates a proper nonzero ideal of $M(E)$.

Corollary 7.2 is well known when E is row-finite, in which case it is a consequence of [9, Theorem 5.3]. It follows that $M(E)$ is simple if and only if E is cofinal, meaning that every vertex connects to any infinite path and to any sink [11, Lemma 2.8]. We develop here a suitable version of this criterion for $M(E, C, S)$, provided Assumption (*) of Definitions 5.2, 5.12 holds.

We will only need to consider infinite *forward* paths, that is, paths (e_1, e_2, \dots) with $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. For any finite or infinite path γ and any integer $n \geq 0$, let us write $\gamma[n]$ to denote the n -truncation of γ , that is, the initial subpath consisting of the first n edges of γ . (This is defined only if γ has length at least n .)

It is convenient to introduce a further definition. A vertex v in E is said to be a C_{fin} -sink in case $C_{v,\text{fin}} = \emptyset$.

Definition 7.4. Let (E, C) be a separated graph and let v be a vertex in E . An *infinite C -multipath in E starting at v* is a nonempty collection Γ of paths in E satisfying the following two conditions:

- (a) Every path in Γ starts at v and either is an infinite path or ends in a C_{fin} -sink.
- (b) For any integer $n \geq 0$ and any $\gamma \in \Gamma$ of length at least n , either $r(\gamma[n])$ is a C_{fin} -sink or for every X in $C_{r(\gamma[n]),\text{fin}}$ there exist $f \in X$ and $\gamma' \in \Gamma$ such that $\gamma'[n+1] = \gamma[n]f$.

Similarly, for $m \in \mathbb{Z}^+$, a *C -multipath in E of length m starting at v* is a nonempty collection Γ of finite paths in E satisfying

- (a') Every path in Γ starts at v and either has length exactly m or ends in a C_{fin} -sink.
- (b') Condition (b), for all $n < m$.

In either case, we denote by Γ^0 (respectively, Γ^1) the set of all vertices (respectively, edges) occurring in the paths in Γ .

Let us say that E is *C -cofinal* if, given any vertex $w \in E^0$ and any infinite C -multipath Γ in E , there is a path from w to some vertex in Γ^0 . In particular, this condition implies that there are paths in E from any vertex to any C_{fin} -sink.

In the absence of condition $(*)$ (or some similar property), it is easily possible for $M(E, C)$ to be simple without E being C -cofinal. For example, let E be the graph

$$x \xleftarrow{e} v \xrightarrow{f} y$$

and take $C = S = C_v := \{\{e\}, \{f\}\}$. Then $M(E, C) \cong \mathbb{Z}^+$, a simple monoid. On the other hand, there is no path in E from the vertex x to the C_{fin} -sink y , so E is not C -cofinal.

Lemma 7.5. *Assume condition $(*)$ for (E, C) holds. Let H be a hereditary subset of E^0 , and define $\overline{H} = \bigcup_{n=0}^{\infty} H_n$ where $H_0 := H$ and*

$$H_n := H_{n-1} \cup \{v \in E^0 \mid \text{there exists } X \in C_{v, \text{fin}} \text{ such that } r(X) \subseteq H_{n-1}\}$$

for $n > 0$. Then \overline{H} is hereditary and C -saturated.

Proof. It is clear from the construction of \overline{H} that this set is C -saturated.

In order to make use of condition $(*)$, we require the free abelian monoid F of Section 5, where we take $S = C_{\text{fin}}$. For $\alpha \in F$, we shall denote by $\text{supp}_{E^0}(\alpha)$ the set $E^0 \cap \text{supp}(\alpha)$. The definition of the sets H_n can be rewritten in the form

$$H_n = H_{n-1} \cup \{v \in E^0 \mid \text{supp}_{E^0}(\mathbf{r}(X)) \subseteq H_{n-1} \text{ for some } X \in C_{v, \text{fin}}\}.$$

We show by induction that each H_n is hereditary. If $n = 0$, this is our hypothesis. Now assume that H_{n-1} is hereditary, for some $n \geq 1$. To see that H_n is hereditary, it suffices to show that $r(e) \in H_n$ for every $v \in H_n \setminus H_{n-1}$ and $e \in s^{-1}(v)$. We choose $Z \in \mathcal{Z}_v$ such that $e \in Z$. On the other hand, by definition of H_n there is some $X \in C_{v, \text{fin}}$ such that $r(X) \subseteq H_{n-1}$.

By $(*)$, there is some $\alpha \in F$ such that $\mathbf{r}(X) \rightsquigarrow_1 \alpha$ and $\boldsymbol{\rho}(Z) \rightsquigarrow_1 \alpha$. Since H_{n-1} is hereditary, we see that $\text{supp}_{E^0}(\alpha) \subseteq H_{n-1}$. Now there are various possibilities for $r(e)$. If $r(e) \in \text{supp}_{E^0}(\alpha)$ then $r(e) \in H_{n-1} \subseteq H_n$. If $\mathbf{r}(Y') \leq \alpha$ for some $Y' \in C_{r(e), \text{fin}}$, then $r(e) \in H_n$ too. Finally, if $\boldsymbol{\rho}(Z') \leq \alpha$ for some $Z' \in \mathcal{Z}_{r(e)} \setminus C_{\text{fin}}$, then $q'_{Z'} \leq \alpha$. Since $\mathbf{r}(X) \rightsquigarrow_1 \alpha$, it follows that $r(e) \leq \mathbf{r}(X)$ and so $r(e) \in H_{n-1} \subseteq H_n$. In any case we get $r(e) \in H_n$, as desired. \square

Theorem 7.6. *Let (E, C) be a separated graph and S a subset of C_{fin} . Assume condition $(*)$ for (E, C) holds. Then $M(E, C, S)$ is simple if and only if the following conditions hold:*

- (a) $S = C_{\text{fin}}$.
- (b) E is C -cofinal.

Proof. In view of Theorem 7.1, we may assume that $S = C_{\text{fin}}$, and it suffices to show that E is C -cofinal if and only if the only hereditary C -saturated subsets of E^0 are \emptyset and E^0 .

Assume first that there is a proper nonempty hereditary C -saturated subset H in E^0 . Let $v \in E^0 \setminus H$, and let Γ_0 be the set consisting of the path of length zero at v . Thus, Γ_0 is a C -multipath of length 0 starting at v . Now suppose that, for some $m \in \mathbb{Z}^+$, we have constructed a C -multipath Γ_m of length m starting at v , such that Γ_m^0 is disjoint from H . For each $\gamma \in \Gamma_m$ such that $r(\gamma)$ is not a C_{fin} -sink, and each $X \in C_{r(\gamma), \text{fin}}$, it follows from the C -saturation of H that there is some $f_{m+1} \in X$ such that $r(f_{m+1}) \notin H$. So, we enlarge γ to a path $\gamma' = \gamma f_{m+1}$ of length $m + 1$. The set of paths obtained in this way, for all $\gamma \in \Gamma_m$

and all $X \in C_{r(\gamma), \text{fin}}$, together with the set of paths in Γ_m that end in C_{fin} -sinks, forms a C -multipath Γ_{m+1} of length $m+1$ starting at v , such that Γ_{m+1}^0 is disjoint from H . Thus, we can build C -multipaths Γ_m of length m for every $m \in \mathbb{Z}^+$ in a compatible way. We now define Γ as the set of paths from $\bigcup_{m \geq 0} \Gamma_m$ ending in C_{fin} -sinks, together with those infinite paths γ such that $\gamma[m] \in \Gamma_m$ for all $m \in \mathbb{Z}^+$. Clearly, Γ is an infinite C -multipath, and Γ^0 is disjoint from H . Since H is hereditary, this means that there is no path from any vertex of H to any vertex in Γ^0 . Since H is nonempty, we conclude that E is not C -cofinal.

Conversely, assume that Γ is an infinite C -multipath in E and that w is a vertex of E not connecting to Γ^0 . Let H be the hereditary subset of E^0 generated by w , and form \overline{H} as in Lemma 7.5. Clearly, $H_0 = H$ is disjoint from Γ^0 . Now assume, for some $n \geq 0$, that H_n is disjoint from Γ^0 , and suppose that $v \in H_{n+1} \cap \Gamma^0$. Then by definition, there exists $X \in C_{v, \text{fin}}$ such that $r(X) \subseteq H_n$. On the other hand, there is a path $\gamma \in \Gamma$ such that $v = r(\gamma[m])$ for some m . By the definition of a C -multipath, it follows that there are a path γ' in Γ and an edge $f \in X$ such that $\gamma'[m+1] = \gamma[m]f$. In particular, $r(f) \in H_n \cap \Gamma^0$, a contradiction. Thus, $\overline{H} \cap \Gamma^0 = \emptyset$. Since \overline{H} is hereditary and C -saturated, we conclude that there is a nonempty, proper hereditary C -saturated subset in E^0 . \square

It is interesting to compare our situation with the one in [11, Lemma 2.7]: If E is row-finite, $M(E)$ is simple, and E contains a sink, then E^0 contains a unique sink, and there are no infinite paths in E . In our case, if $M(E, C)$ is simple, then there is at most one sink in E if Assumption $(*)$ holds, but not otherwise (see the example following Definition 7.4). Further, even if $(*)$ holds, infinite paths may occur, as the example in Remark 7.3 shows. Moreover, E may contain arbitrarily many C_{fin} -sinks. For example, let E^0 be an arbitrary nonempty set, choose E^1 to contain infinitely many edges from any vertex in E^0 to any other, and set $C_v = \{s^{-1}(v)\}$ for $v \in E^0$. Then all vertices of E are C_{fin} -sinks, and $(*)$ holds vacuously. Since E is clearly cofinal, $M(E, C) = M(E)$ is simple.

8. RESOLUTIONS

Our next aim is to develop a construction that allows us to embed graph monoids without refinement into ones with refinement. As Wehrung has proved in [40, Proposition 1.5 and Theorem 1.8], every conical abelian monoid can be embedded in a conical refinement monoid. We obtain a more “visual” version of this result, in that all the monoids that appear are graph monoids. We restrict attention to finitely separated graphs since arbitrary conical abelian monoids can be obtained as graph monoids of finitely separated graphs (Proposition 4.4). Our construction process involves adjoining vertices and edges designed to satisfy Assumption $(*)$. We also want the resulting monoid embeddings to preserve properties such as failure of cancellation or separativity. To obtain this, we arrange for embeddings of the following type.

Definition 8.1. Following [40], a monoid homomorphism $\psi : M \rightarrow F$ is *unitary* provided

- (1) ψ is injective;
- (2) $\psi(M)$ is *cofinal* in F , that is, for each $u \in F$ there is some $v \in M$ with $u \leq \psi(v)$;
- (3) whenever $u, u' \in M$ and $v \in F$ with $\psi(u) + v = \psi(u')$, we have $v \in \psi(M)$.

Lemma 8.2. *Let F be the free abelian monoid generated by elements a_{ij} for $1 \leq i \leq n$, $1 \leq j \leq m$. Let (δ_{ij}) be an $n \times m$ matrix of positive integers. Consider the elements $c_i = \sum_{j=1}^m \delta_{ij} a_{ij}$ for $i = 1, \dots, n$, and $d_j = \sum_{i=1}^n \delta_{ij} a_{ij}$ for $j = 1, \dots, m$. Let M be the monoid with generators $x_1, \dots, x_n, y_1, \dots, y_m$, subject to the single relation*

$$\sum_{i=1}^n x_i = \sum_{j=1}^m y_j.$$

Then the natural monoid homomorphism $\psi: M \rightarrow F$ sending x_i to c_i and y_j to d_j is unitary.

Proof. Since all $\delta_{ij} > 0$, we have $a_{ij} \leq c_i = \psi(x_i)$ for all i, j . Therefore $\psi(M)$ is cofinal in F .

Suppose that $u, u' \in M$ and $v \in F$ with $\psi(u) + v = \psi(u')$. Write $u = \sum_{i=1}^n \lambda_i x_i + \sum_{j=1}^m \mu_j y_j$ and $u' = \sum_{i=1}^n \lambda'_i x_i + \sum_{j=1}^m \mu'_j y_j$ for some $\lambda_i, \mu_j, \lambda'_i, \mu'_j \in \mathbb{Z}^+$. Then $\psi(u) = \sum_{i,j} (\lambda_i + \mu_j) \delta_{ij} a_{ij}$ and $\psi(u') = \sum_{i,j} (\lambda'_i + \mu'_j) \delta_{ij} a_{ij}$. Hence, $\lambda_i + \mu_j \leq \lambda'_i + \mu'_j$ for all i, j , and

$$v = \sum_{i,j} (\lambda'_i + \mu'_j - \lambda_i - \mu_j) \delta_{ij} a_{ij}.$$

Since $\lambda_i - \lambda'_i \leq \mu'_j - \mu_j$ for all i, j , there exists $\lambda \in \mathbb{Z}$ such that $\lambda_i - \lambda'_i \leq \lambda \leq \mu'_j - \mu_j$ for all i, j . Then we can define

$$w = \sum_{i=1}^n (\lambda'_i + \lambda - \lambda_i) x_i + \sum_{j=1}^m (\mu'_j - \mu_j - \lambda) y_j \in M,$$

and $\psi(w) = v$. This verifies the third unitarity condition, and it only remains to show that ψ is injective.

It is clear that each element x in M can be written in the form

$$x = \sum_{i=1}^n \lambda_i x_i + \sum_{j=1}^m \mu_j y_j$$

where at least one of the μ_j 's is 0. So it suffices to show that the elements $\sum_{i=1}^n \lambda_i c_i + \sum_{j=1}^m \mu_j d_j$ with at least one $\mu_j = 0$ are all distinct in F .

Assume that

$$\sum_{i=1}^n \lambda_i c_i + \sum_{j=1}^m \mu_j d_j = \sum_{i=1}^n \lambda'_i c_i + \sum_{j=1}^m \mu'_j d_j$$

in F , where there are j_0 and j_1 such that $\mu_{j_0} = 0 = \mu'_{j_1}$. We have

$$\sum_{i=1}^n \sum_{j=1}^m (\lambda_i + \mu_j) \delta_{ij} a_{ij} = \sum_{i=1}^n \sum_{j=1}^m (\lambda'_i + \mu'_j) \delta_{ij} a_{ij},$$

so that $\lambda_i + \mu_j = \lambda'_i + \mu'_j$ for all i, j . Assume first that $j_0 = j_1$. Then we get $\lambda_i = \lambda'_i$ for all i , and then $\mu_j = \mu'_j$ for all j , as desired. If $j_0 \neq j_1$, then we get

$$\lambda_i = \lambda'_i + \mu'_{j_0} = \lambda_i + \mu'_{j_0} + \mu_{j_1},$$

which implies that $\mu'_{j_0} = \mu_{j_1} = 0$, and we reduce to the above case. \square

Corollary 8.3. *Let F be the free abelian monoid generated by elements a_{ij}^γ for $1 \leq i \leq n_\gamma$, $1 \leq j \leq m_\gamma$, $\gamma \in \Gamma$, and let (δ_{ij}^γ) , for $\gamma \in \Gamma$, be corresponding matrices of positive integers. Consider the elements $c_i^\gamma = \sum_{j=1}^{m_\gamma} \delta_{ij}^\gamma a_{ij}^\gamma$ for $i = 1, \dots, n_\gamma$ and $d_j^\gamma = \sum_{i=1}^{n_\gamma} \delta_{ij}^\gamma a_{ij}^\gamma$ for $j = 1, \dots, m_\gamma$. Let M be the monoid given by generators x_i^γ and y_j^γ for all i, j, γ , subject to the relations*

$$\sum_{i=1}^{n_\gamma} x_i^\gamma = \sum_{j=1}^{m_\gamma} y_j^\gamma, \quad (\gamma \in \Gamma),$$

and let $\psi : M \rightarrow F$ be the natural homomorphism sending $x_i^\gamma \mapsto c_i^\gamma$ and $y_j^\gamma \mapsto d_j^\gamma$ for all i, j, γ . Then ψ is unitary.

Proof. Observe that $F = \bigoplus_{\gamma \in \Gamma} F_\gamma$ where F_γ is the free abelian monoid generated by the a_{ij}^γ for $1 \leq i \leq n_\gamma$ and $1 \leq j \leq m_\gamma$, and $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ where M_γ is the monoid given by generators $x_1^\gamma, \dots, x_{n_\gamma}^\gamma, y_1^\gamma, \dots, y_{m_\gamma}^\gamma$ and the relation $\sum_{i=1}^{n_\gamma} x_i^\gamma = \sum_{j=1}^{m_\gamma} y_j^\gamma$. Further, $\psi = \bigoplus_{\gamma \in \Gamma} \psi_\gamma$ where $\psi_\gamma : M_\gamma \rightarrow F_\gamma$ is the natural homomorphism sending $x_i^\gamma \mapsto c_i^\gamma$ and $y_j^\gamma \mapsto d_j^\gamma$. By Lemma 8.2, each ψ_γ is unitary, and therefore ψ is unitary. \square

Recall from Definitions 3.1 and 3.2 the categories **SSGr** and **SGr**, as well as the definition of an SG -subgraph of a separated graph (E, C) . We shall need a category of finitely separated graphs, defined as follows:

Definition 8.4. Define a category **FSGr** whose objects are all finitely separated graphs (E, C) . A morphism $\phi : (E, C) \rightarrow (F, D)$ in **FSGr** is any graph morphism $\phi : E \rightarrow F$ such that

- (1) ϕ^0 is injective.
- (2) For each $v \in E^0$ and each $X \in C_v$, there is some $Y \in D_{\phi^0(v)}$ such that ϕ^1 induces a bijection $X \rightarrow Y$.

Observe that **FSGr** is a full subcategory of **SGr**. Recall that **SGr** admits arbitrary direct limits (Proposition 3.3), and observe that **FSGr** is closed (in **SGr**) under direct limits. Moreover, every object (E, C) in **FSGr** is the direct limit of the directed system of its finite complete subobjects, as one can see from the proof of Proposition 3.5. It is worth to mention that this latter property does *not* hold in the category **SGr**.

Construction 8.5. Let (E, C) be an object in **FSGr**, and let $T = \{(w_k, X_k, Y_k) \mid k \in I\}$ be a collection of distinct ordered triples such that for each $k \in I$, w_k is a vertex of E and X_k, Y_k are distinct members of C_{w_k} . Let $\delta = \{\delta^k \mid k \in I\}$, where δ^k is an $X_k \times Y_k$ matrix of positive integers for all k . We construct a δ - T -resolution for (E, C) as follows. It is a finitely separated graph (E_T, C^T) , containing (E, C) as an SG -subgraph, with the following data:

- (1) $E_T^0 := E^0 \sqcup \{v_{e,f}^k \mid k \in I, e \in X_k, f \in Y_k\}$.
- (2) $E_T^1 := E^1 \sqcup \{g_{e,f,j}^k, h_{e,f,j}^k \mid k \in I, e \in X_k, f \in Y_k, 1 \leq j \leq \delta_{e,f}^k\}$.
- (3) $s(g_{e,f,j}^k) = r(e)$, $s(h_{e,f,j}^k) = r(f)$ and $r(g_{e,f,j}^k) = r(h_{e,f,j}^k) = v_{e,f}^k$ for all $k \in I, e \in X_k, f \in Y_k, 1 \leq j \leq \delta_{e,f}^k$.

- (4) $X_e^k := \{g_{e,f,j}^k \mid f \in Y_k, 1 \leq j \leq \delta_{e,f}^k\}$ for $k \in I$, $e \in X_k$, and
 $Y_f^k := \{h_{e,f,j}^k \mid e \in X_k, 1 \leq j \leq \delta_{e,f}^k\}$ for $k \in I$, $f \in Y_k$.
- (5) $C_u^T := C_u \sqcup \{X_e^k \mid k \in I, e \in X_k, r(e) = u\} \sqcup \{Y_f^k \mid k \in I, f \in Y_k, r(f) = u\}$ for all $u \in E^0$.
- (6) $C_u^T := \emptyset$ for all $u \in E_T^0 \setminus E^0$.

Observe that in the free abelian monoid on E_T^0 , we have $r(e) \rightarrow_1 \mathbf{r}(X_e^k) = \sum_{f \in Y_k} \delta_{e,f}^k v_{e,f}^k$ for all $k \in I$, $e \in X_k$, and so

$$\mathbf{r}(X_k) \rightsquigarrow_1 \sum_{e \in X_k, f \in Y_k} \delta_{e,f}^k v_{e,f}^k$$

for all k . Similarly, $\mathbf{r}(Y_k) \rightsquigarrow_1 \sum_{e \in X_k, f \in Y_k} \delta_{e,f}^k v_{e,f}^k$ for all k , and thus $(*)$ holds in (E_T, C^T) for w_k, X_k, Y_k .

Lemma 8.6. *Let (E, C) be a separated graph, and T, δ as in Construction 8.5. Let (E_T, C^T) be a δ - T -resolution for (E, C) , and $\iota : (E, C) \rightarrow (E_T, C^T)$ the inclusion morphism. Then $M(\iota) : M(E, C) \rightarrow M(E_T, C^T)$ is unitary.*

Proof. Keep the notation of Construction 8.5, and set $\mu := M(\iota)$.

Let F be the free abelian monoid with generators $a_{e,f}^k$ for $k \in I$, $e \in X_k$, $f \in Y_k$, and let M be the monoid given by generators x_e^k for $k \in I$, $e \in X_k$ and y_f^k for $k \in I$, $f \in Y_k$ subject to the relations $\sum_{e \in X_k} x_e^k = \sum_{f \in Y_k} y_f^k$ for $k \in I$. There is a natural homomorphism $\psi : M \rightarrow F$ sending $x_e^k \mapsto \sum_{f \in Y_k} \delta_{e,f}^k a_{e,f}^k$ for $k \in I$, $e \in X_k$ and $y_f^k \mapsto \sum_{e \in X_k} \delta_{e,f}^k a_{e,f}^k$ for $k \in I$, $f \in Y_k$, and Corollary 8.3 shows that ψ is unitary.

There is a unique homomorphism $\eta : M \rightarrow M(E, C)$ sending $x_e^k \mapsto r(e)$ for $k \in I$, $e \in X_k$ and $y_f^k \mapsto r(f)$ for $k \in I$, $f \in Y_k$ (because $\mathbf{r}(X_k) = w_k = \mathbf{r}(Y_k)$ in $M(E, C)$), and there is a unique homomorphism $\eta' : F \rightarrow M(E_T, C^T)$ sending $a_{e,f}^k \mapsto v_{e,f}^k$ for $k \in I$, $e \in X_k$, $f \in Y_k$. We observe that the following diagram commutes:

$$(8.1) \quad \begin{array}{ccc} M & \xrightarrow{\psi} & F \\ \eta \downarrow & & \downarrow \eta' \\ M(E, C) & \xrightarrow{\mu} & M(E_T, C^T) \end{array}$$

This holds because

$$\mu\eta(x_e^k) = r(e) = \mathbf{r}(X_e^k) = \sum_{f \in Y_k} \delta_{e,f}^k v_{e,f}^k = \sum_{f \in Y_k} \delta_{e,f}^k \eta'(a_{e,f}^k) = \eta'\psi(x_e^k)$$

for all $k \in I$, $e \in X_k$, and similarly $\mu\eta(y_f^k) = \eta'\psi(y_f^k)$ for all $k \in I$, $f \in Y_k$. We claim that (8.1) is a pushout in the category of abelian monoids. It will then follow from [40, Lemma 1.6] that μ is unitary, completing the proof.

Suppose we are given a monoid N and homomorphisms $\sigma : M(E, C) \rightarrow N$ and $\sigma' : F \rightarrow N$ such that $\sigma\eta = \sigma'\psi$. Set $n_u := \sigma(u)$ for $u \in E^0$ and $b_{e,f}^k := \sigma'(a_{e,f}^k)$ for $k \in I$, $e \in X_k$,

$f \in Y_k$. Let F^T denote the free abelian monoid on E_T^0 , and identify $M(E_T, C^T)$ with F^T / \sim as in Section 5. Then let $\tau : F^T \rightarrow N$ be the unique homomorphism such that $\tau(u) = n_u$ for $u \in E^0$ and $\tau(v_{e,f}^k) = b_{e,f}^k$ for $k \in I$, $e \in X_k$, $f \in Y_k$.

In order to see that τ factors through $M(E_T, C^T)$, we need to show that

$$(8.2) \quad \tau(u) = \tau(\mathbf{r}(Z))$$

for any $u \in E_T^0$ and any $Z \in C_u^T$. Since $C_u^T = \emptyset$ for all $u \in E_T^0 \setminus E^0$, we only need to consider $u \in E^0$. There are two cases, depending on whether or not $Z \in C_u$.

Assume first that $Z \in C_u$. Then $u = \mathbf{r}(Z)$ in $M(E, C)$, and since the ranges of the edges in Z all lie in E^0 , we have

$$\tau(u) = n_u = \sigma(u) = \sigma(\mathbf{r}(Z)) = \sum_{d \in Z} \sigma(r(d)) = \sum_{d \in Z} n_{r(d)} = \sum_{d \in Z} \tau(r(d)) = \tau(\mathbf{r}(Z)).$$

This verifies (8.2) when $Z \in C_u$. If $Z \notin C_u$, then either $Z = X_e^k$ for some $k \in I$, $e \in X_k$ with $r(e) = u$, or $Z = Y_f^k$ for some $k \in I$, $f \in Y_k$ with $r(f) = u$. These two cases are symmetric; we treat the former. Then,

$$\begin{aligned} \tau(u) &= n_u = \sigma(r(e)) = \sigma\eta(x_e^k) = \sigma'\psi(x_e^k) \\ &= \sum_{f \in Y_k} \delta_{e,f}^k \sigma'(a_{e,f}^k) = \sum_{f \in Y_k} \delta_{e,f}^k b_{e,f}^k = \sum_{f \in Y_k} \delta_{e,f}^k \tau(v_{e,f}^k) = \tau(\mathbf{r}(X_e^k)). \end{aligned}$$

This verifies (8.2) in the case $Z = X_e^k$.

Now τ induces a homomorphism $\bar{\tau} : M(E_T, C^T) \rightarrow N$. In particular, $\bar{\tau}(u) = n_u = \sigma(u)$ for all $u \in E^0$, which implies that $\bar{\tau}\mu = \sigma$. Also, $\bar{\tau}\eta'(a_{e,f}^k) = \bar{\tau}(v_{e,f}^k) = b_{e,f}^k = \sigma'(a_{e,f}^k)$ for $k \in I$, $e \in X_k$ and $f \in Y_k$, whence $\bar{\tau}\eta' = \sigma'$. Uniqueness of $\bar{\tau}$ with respect to these equations is clear, and therefore (8.1) is a pushout, as desired. \square

Remark 8.7. (Universal property) Given a set I and an (abelian) monoid N , consider a set \mathcal{E} of equations in N :

$$\sum_{n=1}^{N_k} a_n^k = \sum_{m=1}^{M_k} b_m^k \quad (k \in I).$$

Let $\delta = \{\delta^k \mid k \in I\}$ be a set of integer matrices with strictly positive entries, where each δ^k is $N_k \times M_k$. A δ -refinement of \mathcal{E} in N is a set of elements

$$\{c_{n,m}^k \mid k \in I, 1 \leq n \leq N_k, 1 \leq m \leq M_k\} \subseteq N$$

such that $a_n^k = \sum_{m=1}^{M_k} \delta_{n,m}^k c_{n,m}^k$ and $b_m^k = \sum_{n=1}^{N_k} \delta_{n,m}^k c_{n,m}^k$ for all k, n, m .

The pushout property appearing in the proof of the above lemma is equivalent to the following universal property of $M(E_T, C^T)$: Given a monoid homomorphism $\Phi : M(E, C) \rightarrow N$, given a family $T = \{(w_k, X_k, Y_k) \mid k \in I\}$ as in Construction 8.5 with $N_k = |X_k|$ and $M_k = |Y_k|$ for all $k \in I$, and given a δ -refinement

$$\{c_{e,f}^k \mid k \in I, e \in X_k, f \in Y_k\}$$

of the set of equations $\Phi(\mathbf{r}(X_k)) = \Phi(\mathbf{r}(Y_k))$, $k \in I$, in N , there exists a unique monoid homomorphism $\tilde{\Phi}: M(E_T, C^T) \rightarrow N$ such that $\tilde{\Phi}(v) = \Phi(v)$ for all $v \in E^0$ and $\tilde{\Phi}(v_{e,f}^k) = c_{e,f}^k$ for all $k \in I$, $e \in X_k$, $f \in Y_k$.

Construction 8.8. Let (E, C) be an object in **FSGr**. Recursively, construct a δ_n - T_n -resolution (E_{n+1}, C^{n+1}) for (E_n, C^n) , for $n = 0, 1, 2, \dots$, where

- (1) $(E_0, C^0) := (E, C)$.
- (2) For each $v \in E_n^0 \setminus E_{n-1}^0$ (where $E_{-1}^0 := \emptyset$) and distinct $X, Y \in C_v^n$, at least one of the triples (v, X, Y) or (v, Y, X) appears in T_n . (We explicitly allow the possibility that both of these triples appear. It could be useful to exclude triples for which $(*)$ already holds in (E_n, C^n) .)
- (3) δ_n is a set of matrices of positive integers corresponding to the triples in T_n as in Construction 8.5.

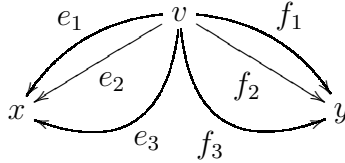
The direct limit of the sequence $(E_0, C^0) \rightarrow (E_1, C^1) \rightarrow \dots$ will be called a *complete resolution* of (E, C) .

Theorem 8.9. Let (E, C) be a finitely separated graph, (E_+, C^+) a complete resolution for (E, C) , and $\iota: (E, C) \rightarrow (E_+, C^+)$ the inclusion morphism. Then $M(E_+, C^+)$ is a refinement monoid and $M(\iota): M(E, C) \rightarrow M(E_+, C^+)$ is unitary.

Proof. It follows immediately from Lemma 8.6 and Construction 8.8 that $M(\iota)$ is unitary. If $v \in E_+^0$ and X, Y are distinct members of C_v^+ , then for some n , either (v, X, Y) or (v, Y, X) appears in T_n . By construction, axiom $(*)$ holds for v, X, Y in (E_{n+1}, C^{n+1}) , and hence also in (E_+, C^+) . Therefore Proposition 5.9 shows that $M(E, C)$ is a refinement monoid. \square

Theorem 8.9 provides a way to construct explicit examples of refinement monoids with particular properties. We illustrate this by constructing a conical refinement monoid which fails to have *separative cancellation*, that is, it contains elements x and y satisfying $2x = x + y = 2y$ but $x \neq y$. The existence of such monoids was first obtained by Wehrung, as a consequence of [40, Corollary 2.7].

Example 8.10. Let E be the graph below



and define $C := \{\{e_1, e_2\}, \{f_1, f_2\}, \{e_3, f_3\}\}$. Then $M(E, C)$ is presented by the generators v, x, y and the relations

$$v = x + x \qquad v = y + y \qquad v = x + y.$$

Observe that $x \neq y$, so that $M(E, C)$ is not separative.

Now choose a complete resolution (E_+, C^+) for (E, C) . No special properties are required of the matrices in the sets δ_n , so they may be chosen with all entries equal to 1. (We

leave to the reader the task of choosing suitable notation for the vertices and edges of E_+ .) By Theorem 8.9, $M(E_+, C^+)$ is a conical refinement monoid, and the inclusion morphism $(E, C) \rightarrow (E_+, C^+)$ induces a unitary embedding $M(E, C) \rightarrow M(E_+, C^+)$, which we denote $a \mapsto a'$. Hence, $2x' = x' + y' = 2y'$ in $M(E_+, C^+)$, but $x' \neq y'$. Therefore $M(E_+, C^+)$ is not separative.

9. REFINING LEAVITT PATH ALGEBRAS OF SEPARATED GRAPHS

We may apply the resolution process of the previous section to *embed* Leavitt path algebras of separated graphs into others with refinement.

Theorem 9.1. *Let (E_+, C^+) be a complete resolution of an object (E, C) in \mathbf{FSGr} , and let $\iota : (E, C) \rightarrow (E_+, C^+)$ be the inclusion morphism. Then $\mathcal{V}(L_K(E_+, C^+)) \cong M(E_+, C^+)$ is a refinement monoid, the K -algebra homomorphism $L_K(\iota) : L_K(E, C) \rightarrow L_K(E_+, C^+)$ is an embedding, and the monoid homomorphism*

$$\mathcal{V}(L_K(\iota)) : \mathcal{V}(L_K(E, C)) \longrightarrow \mathcal{V}(L_K(E_+, C^+))$$

is a unitary embedding.

Proof. The first and third conclusions follow from Theorems 4.3 and 8.9. Since (E, C) is a complete SG -subgraph of (E_+, C^+) , it follows from (the proof of) Proposition 3.6 that the homomorphism $L_K(\iota)$ is injective. \square

As a specific application of our construction, we can unitarily embed any simple conical monoid M into a simple refinement monoid M^+ in such a way that M^+ corresponds to the set of order-units in a monoid of the form $\mathcal{V}(L_K(E_+, C^+))$ together with $\{0\}$, and such that the latter is a simple, divisible, refinement monoid. The existence of such monoid embeddings has been proved by Wehrung [40, Corollary 2.7]; our construction provides a more “visual” version.

Definition 9.2. A monoid M is *divisible* provided every element $x \in M$ is divisible by every positive integer n , that is, there exist elements $y_n \in M$ such that $ny_n = x$ for all $n \in \mathbb{N}$.

Theorem 9.3. *Let M be a simple, conical, abelian monoid with a given presentation by generators x_j ($j \in J$) and relations r_i ($i \in I$) as in (4.1). Assume that for each $j \in J$, there is some $i \in I$ such that $a_{ij} + b_{ij} > 0$. Let (E, C) be the finitely separated graph associated to the given presentation of M as in the proof of Proposition 4.4.*

Then there is a suitable complete resolution (E_+, C^+) of (E, C) such that $\mathcal{V}(L_K(E_+, C^+))$ is a divisible refinement monoid and $M \cong \mathcal{V}(L_K(E, C))$ unitarily embeds in the simple refinement monoid consisting of the union of $\{0\}$ and the semigroup of order-units of $\mathcal{V}(L_K(E_+, C^+))$. If the given presentation of M is countable (i.e., I and J are countable), then $L_K(E_+, C^+)$ is a countably generated K -algebra.

Remark. The hypothesis concerning the $a_{ij} + b_{ij}$ is harmless, since we can always impose additional relations of the form $x_j = x_j$.

Proof. The assumptions of Proposition 4.4 require that for each $i \in I$, at least one $a_{ij} > 0$ and at least one $b_{ij} > 0$. Hence, each vertex u_i emits at least two edges. Our present hypotheses require that each v_j receives at least one edge. Thus, none of the u_i is a sink in E , and none of the v_j is a source in E .

Build a complete resolution (E_+, C^+) for (E, C) as in Construction 8.8, where each matrix in each δ_n is chosen with all entries equal to $n!$. Further, choose the collections T_n to be *symmetric*, meaning that for all $v \in E_n^0 \setminus E_{n-1}^0$ and distinct $X, Y \in C_v^n$, both (v, X, Y) and (v, Y, X) appear in T_n . Set $A := L_K(E, C)$ and $A^+ := L_K(E_+, C^+)$. By Theorem 9.1, $\mathcal{V}(A^+)$ is a refinement monoid and the monoid homomorphism $\mathcal{V}(L_K(\iota)) : \mathcal{V}(A) \rightarrow \mathcal{V}(A^+)$ is a unitary embedding, where $\iota : (E, C) \rightarrow (E_+, C^+)$ is the inclusion morphism. If I and J are countable, then $E_0 = E$ is a countable graph, all the collections T_n are countable, and all the graphs E_n are countable. Consequently, A^+ is a countably generated K -algebra. Now return to the general case.

Since $\mathcal{V}(A) \cong M$ is conical and simple, each of its nonzero elements is an order-unit. By definition of a unitary embedding, the image of $\mathcal{V}(L_K(\iota))$ is cofinal in $\mathcal{V}(A^+)$, from which it follows that $\mathcal{V}(L_K(\iota))$ maps all nonzero elements of $\mathcal{V}(A)$ to order-units of $\mathcal{V}(A^+)$. Consequently, $\mathcal{V}(A)$ embeds unitarily in the submonoid $\{0\} \sqcup S$ of $\mathcal{V}(A^+)$ by way of $\mathcal{V}(L_K(\iota))$, where S is the semigroup of order-units in $\mathcal{V}(A^+)$. Note that $\{0\} \sqcup S$ is a simple monoid.

We now show that $\mathcal{V}(A^+)$ is divisible. In particular, it is then weakly divisible in the sense of [34, Definition 2.2]. Once this is established, a recent result of Ortega, Perera, and Rørdam [34, Theorem 3.4] will show that $\{0\} \sqcup S$ is a refinement monoid, completing the proof of the theorem.

Claim 1: $|C_v^{n+1}| \geq 2$ for all $v \in E_n^0$. In particular, this will show that E_+ has no sinks.

For any $i \in I$, we have $C_{u_i}^0 = \{X_{i1}, X_{i2}\}$ by construction, and $C_{u_i}^0 \subseteq C_{u_i}^1$, so $|C_{u_i}^1| \geq 2$. For any $j \in J$, there is some $i \in I$ such that $a_{ij} + b_{ij} > 0$ (by hypothesis), so there exists $e \in E_0^1$ with $s(e) = u_i$ and $r(e) = v_j$. Since T_0 is symmetric, it contains both (u_i, X_{i1}, X_{i2}) and (u_i, X_{i2}, X_{i1}) , say labelled (w_k, X_k, Y_k) and (w_l, X_l, Y_l) . After possibly interchanging k and l , we may assume that $e \in X_k = Y_l$. Then we have $X_e^k, Y_e^l \in C_{v_j}^1$, and so $|C_{v_j}^1| \geq 2$. This establishes the claim when $n = 0$.

Now let $n > 0$, and assume the claim holds for $n - 1$, that is, $|C_v^n| \geq 2$ for all $v \in E_{n-1}^0$. For $v \in E_{n-1}^0$, we have $C_v^n \subseteq C_v^{n+1}$, and so $|C_v^{n+1}| \geq |C_v^n| \geq 2$ by our induction hypothesis. Now consider $v \in E_n^0 \setminus E_{n-1}^0$. Then there exist $(w_k, X_k, Y_k) \in T_{n-1}$ and $e \in X_k, f \in Y_k$ such that $v = v_{e,f}^k$. There is an edge $g := g_{e,f,1}^k \in E_n^1$ with $s(g) = r(e)$ and $r(g) = v$, and $g \in X_e^k \in C_{r(e)}^n$. Since $r(e) \in E_{n-1}^0$, the induction hypothesis implies that $C_{r(e)}^n$ contains a set $Z \neq X_e^k$. As in the case of v_j above, it follows that there are distinct sets X_g^k and Y_g^l in C_v^{n+1} . This establishes Claim 1.

Set $M^+ := M(E_+, C^+) \cong \mathcal{V}(A^+)$, and let $m \in \mathbb{N}$. To see that $\mathcal{V}(A^+)$ is m -divisible, it suffices to show that each $v \in E_+^0$ is divisible by m in M^+ . Assume first that $v \in E_n^0 \setminus E_{n-1}^0$ for some $n \geq m$. By Claim 1, there exists a set $X \in C_v^{n+1}$. Note that v must be a sink in E_n , so $X \notin C_v^n$. Consequently, X must have the form X_e^k or Y_f^k for appropriate k, e, f . Since we have chosen $\delta_{e,f}^{n,k} = n!$ for all k, e, f , the number n_w of edges in X from v to any

vertex $w \in E_{n+1}^0$ is divisible by m . In M^+ , we have $v = \mathbf{r}(X) = \sum_{w \in E_{n+1}^0} n_w w$, and thus v is divisible by m .

Next, assume that $v \in E_n^0 \setminus E_{n-1}^0$ where $0 < n < m$, and that all vertices from $E_{n+1}^0 \setminus E_n^0$ are divisible by m in M^+ . By Claim 1, there exists a set $X \in C_v^{n+1}$; as above, $X \notin C_v^n$. Then $v = \sum_{e \in X} r(e)$ in M^+ , and $r(e)$ is divisible by m for each $e \in X$ (because $r(e) \in E_{n+1}^0 \setminus E_n^0$), whence v is divisible by m . The same argument applies to the vertices v in $\{v_j \mid j \in J\}$. Finally, for $i \in I$ we have $u_i = \mathbf{r}(X_{i1}) = \sum_{j \in J} a_{ij} v_j$, and thus u_i is divisible by m in M^+ . Therefore all the generators of M^+ are divisible by m , and the proof is complete. \square

Example 9.4. Let M be the monoid $\langle x \mid mx = nx \rangle$ presented by one generator x and one relation $mx = nx$, where $1 < m < n$. The separated graph (E, C) associated to this presentation is just the separated graph $(E(m, n), C(m, n))$ of Example 2.11, so that $L_K(E, C) = A_{m, n}$ in the notation of the example. By Proposition 2.12, we can identify $\mathcal{V}(A_{m, n}) = \langle x \mid mx = nx \rangle$ so that $[w] = x$. It is immediate from the properties of this monoid that the idempotent matrices $w, 2 \cdot w, \dots, (m-1) \cdot w$ are finite (i.e., not equivalent to proper subidempotents of themselves), whereas $m \cdot w \sim n \cdot w \sim v$ is properly infinite (i.e., $m \cdot w \oplus m \cdot w \lesssim m \cdot w$).

The hypotheses of Theorem 9.3 are clearly satisfied, so that $A_{m, n}$ is embedded in the countably generated algebra $L_K(E_+, C^+)$ given by the theorem, and $\mathcal{V}(A_{m, n})$ is unitarily embedded in $\mathcal{V}(L_K(E_+, C^+))$. Due to the unitarity of this embedding, the idempotent matrices $w, 2 \cdot w, \dots, (m-1) \cdot w$ remain finite (and full) over $L_K(E_+, C^+)$, while obviously $m \cdot w$ remains properly infinite. This information is faithfully recorded in $\mathcal{V}(L_K(E_+, C^+))$, which by the theorem is a divisible refinement monoid.

10. APPENDIX. PROJECTIVE MODULES AND TRACE IDEALS FOR NONUNITAL RINGS

In this appendix, we gather some information and results concerning projective modules, trace ideals, and \mathcal{V} -monoids for nonunital rings. Some of this material is well known, but some is not readily accessible, and some has not been developed in the literature to our knowledge.

Definition 10.1. A ring R is *idempotent* provided $R^2 = R$, and it is *s-unital* if for each $x \in R$, there exist $u, v \in R$ such that $ux = x = xv$. The latter property carries over to finite sets by [3, Lemma 2.2]: If R is s-unital and $x_1, \dots, x_n \in R$, there exists $u \in R$ such that $ux_i = x_i = x_i u$ for all i .

As is common (see [25], for instance), we define $R\text{-Mod}$ to be the category of those left R -modules which are *full* (meaning that $RM = M$) and *nondegenerate* (meaning that $Rx = 0$ implies $x = 0$, for any $x \in M$). The morphisms in $R\text{-Mod}$ are arbitrary module homomorphisms between the above modules. We refer to R itself as *nondegenerate* if it is nondegenerate as both a left and a right R -module.

Note that if R is s-unital, then so is any full left R -module M : for any $y_1, \dots, y_m \in M$, there is some $u \in R$ such that $uy_j = y_j$ for all j . In particular, it follows that M is nondegenerate. In fact, all R -submodules of M are full and nondegenerate.

Let R^+ denote the canonical unitization of R (irrespective of whether R may have a unit), namely the unital ring containing R as a two-sided ideal such that $R^+ = \mathbb{Z} \oplus R$. The forgetful functor provides a category isomorphism from $R^+\text{-Mod}$ to the category of arbitrary left R -modules [24, Proposition 8.29B]. We identify these two categories, and then $R\text{-Mod}$ becomes identified with the full subcategory of $R^+\text{-Mod}$ whose objects are those R^+ -modules which are full and nondegenerate as R -modules.

Lemma 10.2. *Let R be a nondegenerate idempotent ring. The projective objects in $R\text{-Mod}$ are precisely those which are projective as R^+ -modules, that is, the projective R^+ -modules P such that $RP = P$.*

Proof. As noted in [26, §5.3], all epimorphisms in $R\text{-Mod}$ are surjective. Consequently, any object of $R\text{-Mod}$ which is projective in $R^+\text{-Mod}$ must also be projective in $R\text{-Mod}$. Conversely, suppose P is a projective object of $R\text{-Mod}$. Choose a free R^+ -module F and an R^+ -module epimorphism $f : F \rightarrow P$. Then RF is a full nondegenerate R -module, and f maps RF onto P , so there exists an R -module homomorphism $g : P \rightarrow RF$ such that $fg = \text{id}_P$. Since g is also an R^+ -module homomorphism, P is a projective R^+ -module.

Since R is nondegenerate, all projective R^+ -modules are nondegenerate as R -modules. Hence, any projective R^+ -module P satisfying $RP = P$ is an object of $R\text{-Mod}$, projective by the previous paragraph. The previous discussion also shows that all projective objects of $R\text{-Mod}$ have this form. \square

Definition 10.3. Let us define a ring R to be *left hereditary* if every subobject of a projective object in $R\text{-Mod}$ is projective. We define “right hereditary” symmetrically, and say R is *hereditary* provided it is both left and right hereditary.

Corollary 10.4. *Let R be a nondegenerate idempotent ring.*

- (1) *If R^+ is (left) hereditary, then R is (left) hereditary.*
- (2) *If R is s-unital and (left) hereditary, then eRe is (left) hereditary for all idempotents $e \in R$.*

Proof. (1) This follows from Lemma 10.2.

(2) Assume that R is s-unital and left hereditary, and let $e \in R$ be an idempotent. We note that $Re = R^+e$ and $eRe = eR^+e$. By Lemma 10.2, R^+e is projective in $R\text{-Mod}$. Since R is s-unital, any R^+ -submodule N of R^+e is an object of $R\text{-Mod}$. Then N is projective in $R\text{-Mod}$ by our hypothesis on R , and hence projective as an R^+ -module by Lemma 10.2. Thus, all R^+ -submodules of R^+e are projective, which implies that eR^+e is left hereditary [28, Theorem 2.5] (or see [41, §39.16]). \square

Definition 10.5. If A is a K -algebra, we shall write A^\sim for the canonical K -algebra unitization of A , that is, the unital K -algebra containing A as a two-sided ideal such that $A^\sim = K \oplus A$.

In order to apply the previous results to this setting, we need to be able to identify $A\text{-Mod}$ with a full subcategory of $A^\sim\text{-Mod}$, which requires that all full nondegenerate A -modules are

vector spaces over K in a canonical fashion. It is not clear whether this holds in general, but it does when A is s-unital, as follows.

Lemma 10.6. *Let A be an s-unital K -algebra and M a full left A -module. Then the multiplication map $\mu : A \otimes_{A^+} M \rightarrow M$ is an isomorphism of A -modules. Consequently, there is a unique K -vector space structure on M such that $\alpha(ax) = (\alpha a)x$ for all $\alpha \in K$, $a \in A$, $x \in M$, and thus M has a canonical left A^\sim -module structure.*

Proof. Note: We must use A^+ in the definition of μ , since we do not yet know that M is an A^\sim -module.

We claim there is a map $\lambda : M \rightarrow A \otimes_{A^+} M$ defined as follows: Given $x \in M$, choose $u \in A$ such that $ux = x$, and set $\lambda(x) = u \otimes x$. To see that λ is well defined, suppose that also $u'x = x$ for some $u' \in A$. There exists $v \in A$ such that $vu = u$ and $vu' = u'$, whence

$$u \otimes x = vu \otimes x = v \otimes x = vu' \otimes x = u' \otimes x.$$

A similar computation shows that λ is surjective, and it is clear that $\mu\lambda = \text{id}_M$. Hence, μ is a bijection, and thus an A -module isomorphism.

Since A is a (K, A^+) -bimodule, $A \otimes_{A^+} M$ has a natural K -vector space structure, and this transfers to M via μ . Uniqueness is clear, and the A^\sim -module structure follows. \square

Proposition 10.7. *Let A be an s-unital K -algebra. If A^\sim is (left) hereditary, then A is (left) hereditary.*

Proof. There is a natural ring homomorphism $A^+ \rightarrow A^\sim$, with respect to which the pullback functor embeds $A^\sim\text{-Mod}$ in $A^+\text{-Mod}$. Under this embedding, the A^\sim -modules which are full A -modules correspond precisely to the objects of $A\text{-Mod} \subseteq A^+\text{-Mod}$, in view of Lemma 10.6.

Just as in Lemma 10.2, the projective objects of $A\text{-Mod}$ are precisely those which are projective as A^\sim -modules. The conclusion of the proposition follows. \square

Definition 10.8. Consider again a ring R with unitization R^+ . (Here R^+ could be replaced by any unital ring containing R as a two-sided ideal.) Let $\mathbf{FP}(R, R^+)$ denote the full subcategory of $R^+\text{-Mod}$ whose objects are those finitely generated projective left R^+ -modules P such that $RP = P$. There is a monoid isomorphism from $\mathcal{V}(R)$ onto the Grothendieck monoid of $\mathbf{FP}(R, R^+)$, that is, the monoid of isomorphism classes of objects from $\mathbf{FP}(R, R^+)$, with addition induced from direct sum. The isomorphism sends the class $[e]$ of an idempotent matrix $e \in M_n(R)$ to the isomorphism class of the module $(R^+)^ne$. (For details, see e.g. [26, §5.1].)

If the ring R is idempotent, the objects of $\mathbf{FP}(R, R^+)$ are exactly the compact projective objects of $R\text{-Mod}$ [26, Lemma 5.5].

In general, the *trace* of a left module M over a ring R is the sum of the images of all homomorphisms from M to R ; it is a two-sided ideal of R . If R is unital and M is finitely generated projective, say $M \cong R^ne$ for some idempotent matrix $e \in M_n(R)$, then the trace of M is generated by the entries of e . This leads one to define the *trace ideals* of R as those ideals which can be generated by the entries of idempotent matrices. In the unital case, there

is an isomorphism between the lattice of trace ideals of R and the lattice of order-ideals in $\mathcal{V}(R)$, as follows from [22, Theorem 2.1(c)]. Since the proof is indirect, and does not extend immediately to the non-unital case, we develop the general result here.

Definition 10.9. Let R be an arbitrary ring. Recall the ring $M_\infty(R)$ from §1.2, and let $\text{Idem}(M_\infty(R))$ denote the set of idempotents in $M_\infty(R)$. An ideal I of R is called a *trace ideal* provided I can be generated by the entries of the matrices in some subset of $\text{Idem}(M_\infty(R))$. We denote by $\text{Tr}(R)$ the set of all trace ideals of R . Since $\text{Tr}(R)$ is closed under arbitrary sums, it forms a complete lattice with respect to inclusion.

Recall that an *order-ideal* of a monoid M is a submonoid I of M such that $x + y \in I$ (for some $x, y \in M$) implies that both x and y belong to I . An order-ideal can also be described as a submonoid I of M which is *hereditary* with respect to the canonical pre-order \leq on M , meaning that $x \leq y$ and $y \in I$ imply $x \in I$. Recall that the pre-order \leq on M is defined by setting $x \leq y$ if and only if there exists $z \in M$ such that $y = x + z$.

The family $\mathcal{L}(M)$ of all order-ideals of M is closed under arbitrary intersections, and hence it forms a complete lattice with respect to inclusion. The supremum of a family $\{I_i\}$ of order-ideals of M is the set $\overline{\sum I_i}$ consisting of those elements $x \in M$ such that $x \leq y$ for some y belonging to the algebraic sum $\sum I_i$. Note that $\overline{\sum I_i} = \sum I_i$ whenever M is a refinement monoid.

Proposition 10.10. *For any ring R there are mutually inverse lattice isomorphisms*

$$\Phi : \mathcal{L}(\mathcal{V}(R)) \longrightarrow \text{Tr}(R) \quad \text{and} \quad \Psi : \text{Tr}(R) \longrightarrow \mathcal{L}(\mathcal{V}(R))$$

given by the rules

$$\begin{aligned} \Phi(I) &= \langle \text{entries of } e \mid e \in \text{Idem}(M_\infty(R)) \text{ and } [e] \in I \rangle \\ \Psi(J) &= \{[e] \in \mathcal{V}(R) \mid e \in \text{Idem}(M_\infty(J))\}. \end{aligned}$$

Proof. Clearly, Φ is a well-defined, inclusion-preserving map from $\mathcal{L}(\mathcal{V}(R))$ to $\text{Tr}(R)$. Now let $J \in \text{Tr}(R)$; we need to show that $\Psi(J)$ is an order-ideal in $\mathcal{V}(R)$. First note that if $e, e' \in \text{Idem}(M_\infty(R))$ and $e \sim e'$, then $e \in M_\infty(J)$ if and only if $e' \in M_\infty(J)$. This holds simply because $M_\infty(J)$ is an ideal of $M_\infty(R)$. Hence, we can check whether an element x of $\mathcal{V}(R)$ lies in $\Psi(J)$ by considering any representative of x in $M_\infty(R)$. Now if $[e], [f] \in \Psi(J)$, then $e, f \in M_\infty(J)$ and so $e \oplus f \in M_\infty(J)$, whence $[e] + [f] \in \Psi(J)$. Similarly, if $e \oplus f \in M_\infty(J)$, then both e and f belong to $M_\infty(J)$, so that $[e] + [f] \in \Psi(J)$ implies $[e], [f] \in \Psi(J)$. This shows that $\Psi(J) \in \mathcal{L}(\mathcal{V}(R))$. Therefore Ψ is a well-defined map from $\text{Tr}(R)$ to $\mathcal{L}(\mathcal{V}(R))$. It clearly preserves inclusions.

It remains to show that Φ and Ψ are inverses of each other, for then they will be isomorphisms of posets, and therefore lattice isomorphisms.

First, we observe that $\Phi \circ \Psi = \text{id}_{\text{Tr}(R)}$: If J is a trace ideal of R , then J is generated by the entries of the matrices in $\text{Idem}(M_\infty(J))$, and so $\Phi\Psi(J) = J$.

Finally, we show that $\Psi \circ \Phi = \text{id}_{\mathcal{L}(\mathcal{V}(R))}$. Let $I \in \mathcal{L}(\mathcal{V}(R))$, and set $J := \Phi(I)$. Clearly, $I \subseteq \Psi(J)$. If $[f] \in \Psi(J)$, there are idempotents e_1, \dots, e_m in $M_\infty(R)$, with $[e_l] \in I$ for each l , such that every entry of f has the form $f_{ij} = \sum_{l=1}^m a_{ij}^l u_{ij}^l b_{ij}^l$ where the $a_{ij}^l, b_{ij}^l \in R$ and the u_{ij}^l

are entries of e_l . Now $e := e_1 \oplus \cdots \oplus e_m$ is an idempotent in $M_\infty(R)$ with $[e] \in I$, and each u_{ij}^l is an entry of e . We may treat e as a $t \times t$ matrix, for some t . Each term $a_{ij}^l u_{ij}^l b_{ij}^l$ can be expressed in the form $x_{ij}^l e y_{ij}^l$ where x_{ij}^l is a $1 \times t$ matrix over R and y_{ij}^l is a $t \times 1$ matrix over R . Let e' denote the block diagonal $mt \times mt$ matrix with m copies of e on the diagonal. Then each

$$f_{ij} = \sum_{l=1}^m x_{ij}^l e y_{ij}^l = x'_{ij} e' y'_{ij}$$

for some $1 \times mt$ matrix x'_{ij} and some $mt \times 1$ matrix y'_{ij} . It follows that we can express f as a product of block matrices of the form

$$f = x'' e'' y'' = \begin{bmatrix} x'_{11} & x'_{12} & \cdots & & \\ & & & x'_{21} & x'_{22} & \cdots \\ & & & & & \ddots \\ & & & & & & \ddots \\ & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} e' & & & \\ & e' & & \\ & & \ddots & \\ & & & e' \end{bmatrix} \begin{bmatrix} y'_{11} & & & \\ & y'_{12} & & \\ & & \ddots & \\ y'_{21} & & & \\ & y'_{22} & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix},$$

where e'' is idempotent and $[e''] \in I$. Now $g := e'' y'' f x'' e''$ is an idempotent in $M_\infty(R)$ such that $f \sim g$ and $g \leq e''$. Consequently, $[f] \leq [e'']$, and so $[f] \in I$ because I is an order-ideal of $\mathcal{V}(R)$. Therefore $I = \Psi(J) = \Psi\Phi(I)$, as required. \square

ACKNOWLEDGEMENTS

We thank G. Bergman, T. Katsura, E. Pardo, and the referee for useful comments and correspondence.

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